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ABSTRACT

The teaching of problem solving begins the moment a child first enters school and the senior high school plays a major role in the development of this skill since a number of students terminate their formal education at the end of this period. This book combines suggestions for the teaching of problem solving with activities, problems, and strategy games that students find interesting as they gain valuable experiences in problem solving. Over 120 classroom-tested problems are included. Discussions in this volume include a definition of problem solving, heuristics, and how to teach problem solving. Also provided are collections of strategy games and nonroutine problems, including 35 reproducible blackline masters for selected problems and game boards; and a bibliography of 51 resources on problem solving. (CW)

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PROBLEM SOLVING

A HANDBOOK FOR TEACHERS

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A ready-to-use resource for teaching problem solving in high school mathematics classes—including over 120 classroom-tested problems, 35 reproducible blackline masters, educational games, and illustrative examples.

SE 050 190

PROBLEM SOLVING

A HANDBOOK FOR SENIOR HIGH SCHOOL TEACHERS

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Allyn and Bacon
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Preface

During the past decade, problem solving has become a major focus of the school mathematics curriculum. As we enter the era of technology, it is more important than ever that our students learn how to succeed in resolving problem situations.

This book is designed to help you, the senior high school mathematics teacher—whether you are a novice or experienced—to teach *problem solving*. Although the teaching of problem solving begins when a child first enters school, the senior high school must play a major role in the development of this skill. Indeed, a significant number of students terminate their formal education at the end of senior high school and thus are dependent upon their elementary and secondary school training to cope with the many problems they face every day. Traditionally, the senior high school mathematics program has been oriented toward preparing students to enter colleges or other institutions of higher education. The content has therefore concentrated on the skills and concepts of algebra, geometry, functions, and other mathematical topics. Problem solving has never really been the major focus of these programs, although it should have been according to the *Agenda for Action* published by the National Council of Teachers of Mathematics.

This book combines suggestions for the teaching of problem solving with activities, carefully discussed non-routine problems, and strategy games your students will find interesting as they gain valuable experiences in problem solving. The activities, problems, and games have been gleaned from a variety of sources and have been classroom-tested by practicing teachers. We believe that this is the first time such an extensive set of problems has appeared in a single resource that is specifically designed for the senior high school.

Problem solving is now considered to be a basic skill of mathematics education, but we suggest that it is more than a single skill. Rather, it is a group of discrete skills. In the chapter on pedagogy, the subskills of problem solving are enumerated and then integrated into a teachable process. The chapter features a flowchart that guides students through this vital process. Although there are many publications that deal with the problem-solving process, we believe that this is the first one that focuses on these subskills.

We are confident that this book will prove to be a valuable asset in your efforts to teach problem solving.

S. K. and J. R.

CHAPTER ONE

An Introduction to Problem Solving

WHAT IS A PROBLEM?

Until very recently, a major difficulty in discussing problem solving was a lack of any clear-cut agreement as to what constituted a "problem." This has finally been resolved; most mathematics educators accept the following definition of a problem:

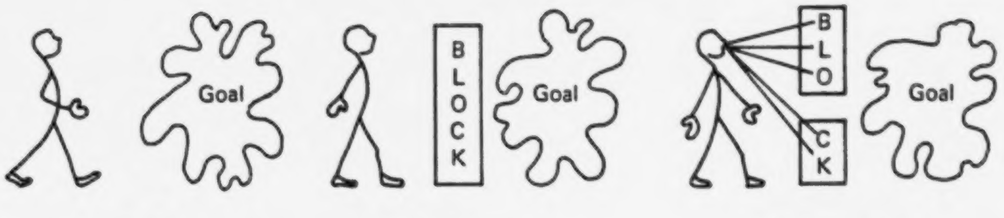
Definition A problem is a situation, quantitative or otherwise, that confronts an individual or group of individuals, that requires resolution, and for which the individual sees no apparent path to the solution.

The key to this definition is the phrase "no apparent path." As children pursue their mathematical training, what were problems at an early age become exercises and eventually reduce to mere questions. We distinguish between these three commonly used terms as follows:

(a) question: a situation that can be resolved by mere *recall and memory*.
(b) exercise: a situation that involves *drill and practice* to reinforce a previously learned skill or algorithm.
(c) problem: a situation that requires *analysis and synthesis* of previously learned knowledge to resolve.

In addition, a problem must be perceived as such by the student, regardless of the reason, in order to be considered a problem by him or her. If the student refuses to accept the challenge, it is not a problem for that student at that time. Thus, a problem must satisfy the following three criteria, illustrated in Figure 1-1.

1. *Acceptance:* The individual accepts the problem. There is personal involvement, which may be due to any of a variety of reasons, including internal motivation, external motivation (peer, parent, and/or teacher pressure), or simply the desire to experience the enjoyment of solving a problem.
2. *Blockage:* The individual's initial attempts at solution are fruitless. His or her habitual responses and patterns of attack do not work.
3. *Exploration:* The personal involvement identified in (1) forces the individual to explore new methods of attack.



Acceptance Blockage Exploration

Figure 1-1

A word about textbook "problems"

The heading "problem" implies that the individual is being confronted by something he or she does not recognize. A situation will no longer be considered a problem once it has been modeled or can easily be solved by applying algorithms that have been previously learned.

While all mathematics textbooks contain sections labeled "word problems," many of these cannot really be considered as problems. In most cases, a model has been developed and a general solution presented in class by the teacher. Following this presentation, the student merely applies the model solution to the subsequent series of exercises in order to solve them. These exercises, except for a change in the numbers and the cast of characters, all fit the same model. Essentially, the student is practicing an algorithm—a technique that applies to a single class of "problems" and that guarantees success if mechanical errors are avoided. Under this format, few of these so-called problems require higher-order thought by the students. Yet the first time a student sees these "word problems," they could be problems to him or her if they are presented in a nonalgorithmic fashion. In many cases, the very placement of these exercises within the text prevents them from being real problems, since they either follow the algorithm designed specifically for their solution or are headed by such statements as "Problem Solving: Time, Rate, and Distance." We consider these sections of the textbook to contain "exercises." Some authors refer to them as "routine problems." We do not advocate removing them from the textbooks, because they do serve a purpose; they provide exposure to problem situations, practice in the use of the algorithm, and drill in the associated mathematical processes. However, a teacher should realize that students who have been solving these exercises through the use of a carefully developed model or algorithm have not been involved in problem solving. In fact, as George Polya stated, solving "the routine problem has practically no chance to contribute to the mental development of the student."*

WHAT IS PROBLEM SOLVING?

Problem solving is a process. It is the means by which an individual uses previously acquired knowledge, skills, and understanding to satisfy the demands of an unfamiliar situation. The process begins with the initial confrontation and concludes when an answer has been obtained and considered with regard to the initial conditions. The student must synthesize what he or she has learned and apply it to the new and different situation.

Some educators assume that expertise in problem solving develops incidentally as one solves many problems. While this may be true in part, we feel that problem solving must be considered as a distinct body of knowledge and that the process should be taught as such.

There are many goals for school mathematics. Two of these are the attainment of information and facts and the ability to use information and facts. The latter ability is an essential part of the problem-solving process. In effect, problem solving requires analysis and synthesis. To succeed in problem solving is to learn how to learn.

WHY A SPECIAL EMPHASIS ON PROBLEM SOLVING?

We believe that a major task of the senior high school mathematics teacher is to provide students with the skills, concepts, and understanding of algebra, geometry, and functions that make up the standard college preparatory program. However, the usual treatment of this material does not provide the student with adequate problem-solving experiences. The major emphasis is upon attaining skills and concepts (power within the subject). Little time is devoted to the development of the open-ended thought process that is problem solving. Achievement in algebra and geometry by itself does not guarantee success in problem solving. Students enrolled in a college preparatory program, as well as those pursuing alternate programs, are required to resolve problems, quantitative or otherwise, every day of their lives. Rarely, if ever, can these problems be resolved by merely referring to a mathematical fact or a previously learned algorithm. The words "Solve me!", "Factor me!", or "Find my area!" never appear in a store window. *Problem solving is the link between facts and algorithms and the real-life problem situations we all face.* For most people, mathematics *is* problem solving!

---

* George Polya. "On Teaching Problem Solving," in *The Role of Axiomatics and Problem Solving in Mathematics*. New York: Ginn, 1966, p. 126.

In spite of the relationship between the mathematics of the classroom and the quantitative situations in life, we know that students see little connection between what happens in school and what happens in real life. An emphasis on problem solving in the classroom can lessen the gap between the real world and the classroom world and thus set a more positive mood in the classroom.

In many mathematics classes, students do not even see any connections among the various ideas taught within a single year. Most regard each topic as a separate entity. Problem solving shows an interconnection between mathematical ideas. Problems are never solved in a vacuum; they are related in some way to something seen before or to something learned earlier. Thus, good problems can be used to review past mathematical ideas, as well as to sow the seeds for ideas to be presented at a future time.

Problem solving is more exciting, more challenging, and more interesting to students than are barren exercises. If we examine student performance in the classroom, we recognize the obvious fact that success leads to persistence and continuation of a task, while failure leads to avoidance. It is this continuation that we constantly strive for in mathematics. The greater the involvement, the better the end product. Thus, a carefully selected sequence of problem-solving activities that yield success will stimulate students, leading them to a more positive attitude towards mathematics in general and problem solving in particular.

Finally, problem solving is an integral part of the larger area of critical thinking, which is a universally accepted goal for all education.

WHEN DO WE TEACH PROBLEM SOLVING?

Problem solving is a skill everyone uses throughout life. The initial teaching and learning of the problem-solving process begins as soon as the child enters school, and it must continue throughout his or her entire school experience. The elementary school teacher has the responsibility for beginning this instruction, laying the foundation for the child's future problem-solving experiences. It remains for the secondary school teacher to build upon this foundation and to complete the formal instruction.

Since the process of problem solving is a teachable skill, when do we teach it? What does it replace? Where does it fit into the day-to-day schedule?

Experiences in problem solving are always at hand. All other activities should be related to problem solving; the teaching of problem solving should be continuous and ongoing. Discussion of problems, proposed solutions, and methods of attacking problems should be considered at all times. Think how poorly students would perform in other skill areas, such as fractions, if they were taught these skills in one or two weeks of concentrated work, after which the skills were never used again.

Naturally, there will be times when studies of algorithmic skills and drill and practice sessions will be called for. We insist that students be competent in the basic skills of arithmetic, algebra, and geometry. Problem solving is not a substitute for these skills. However, these practice sessions allow for the incubation period required by many problems, which need time to "set." By allowing time between problem-solving sessions, you permit students to become familiar with the problem-solving process slowly, and over a longer period of time.

This time is important, since the emphasis is on the process, not merely on obtaining an answer. The development of the process takes time! The number of problems discussed in any one class session must, of necessity, be small. This is a natural outgrowth of the process of problem solving. The goals are a study of the problem-solving process and growth in using this process, rather than merely "covering material."

WHAT MAKES A GOOD PROBLEM SOLVER?

Although we cannot easily determine what makes some students good problem solvers, there are certain common characteristics exhibited by good problem solvers. For instance, good problem solvers know the *anatomy of a problem*. They know that a problem contains facts, a question, and a setting. They also know that most problems (with the exception of some word problems in textbooks) contain distractors, which they can recognize and eliminate.

A good problem solver has a *desire* to solve problems. Problems interest him or her; they offer a challenge. Much like a climber of Mt. Everest, a problem solver likes to solve problems because they exist!

Problem solvers are extremely *perseverant* when solving problems. They are not easily discouraged when incorrect or when a particular approach leads to a dead end. They go back and try new approaches again and again. They refuse to quit! If one method of attacking a problem fails to yield a satisfactory solution, successful problem solvers try another. A variety of methods of attack are usually at their disposal. They will often try the opposite of what they have been doing, in the hope that new information will occur to them. They will ask themselves many "What if . . ." questions, changing conditions within the problem as they proceed.

Good problem solvers show an *ability to skip some of the steps* in the solution process. They make connections quickly, notice irrelevant detail, and often require only a few examples in order to generalize. They often show a marked lack of concern about neatness while developing their solution process.

Above all, good problem solvers are *not afraid to guess*. They will make "educated guesses" at solutions and then attempt to verify these guesses. They will gradually refine their guesses on the basis of what previous guesses have shown them, until they find a satisfactory solution. They rarely guess wildly but use their own intuition to make carefully thought-through guesses.

We would suggest that good problem solvers are students who hold conversations with themselves. They know what questions to ask themselves, and what to do with the answers they receive as they think through the problem.

WHAT MAKES A GOOD PROBLEM?

It should be apparent to the reader that we consider problems to be the basic medium of problem solving. Furthermore, problem solving is the basic skill of mathematics education. It follows, then, that without "good" problems, we could not have creative mathematics.

What constitutes a good problem? We suggest that a good problem contains some or all of the following characteristics:

1. The solution to the problem involves the understanding of distinct mathematical concepts or the use of mathematical skills.

2. The solution of the problem leads to a generalization.
3. The problem is open-ended in that it affords an opportunity for extension.
4. The problem lends itself to a variety of solutions.
5. The problem should be interesting and challenging to the students.

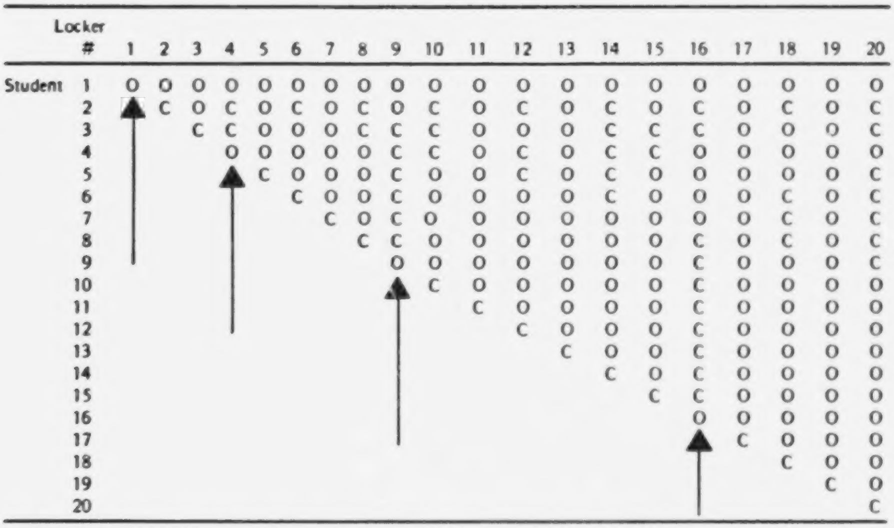
Teachers should be aware that good problems containing these characteristics can be found in every branch of mathematics and in virtually every aspect of daily living. Problems need not be word problems in order to be good problems. What follows in this section are examples that illustrate these characteristics. Keep in mind that not every good problem need have all of these characteristics. Neither is it always possible to identify clearly which characteristic makes a problem good for problem solving—in many cases the characteristics will overlap. However, a good problem will always have some of these attributes.

1. The solution to the problem involves the understanding of distinct mathematical concepts or the use of mathematical skills.

Many problems appear to be non-mathematical in context, yet the solution to the problem involves basic mathematical principles. Perhaps a pattern can be found that the students recognize, or some application of a skill may quickly resolve the problem. In any case, there should be some basic mathematical skill and/or concept embedded in the problem and its solution.

PROBLEM The new school has exactly 1,000 lockers and exactly 1,000 students. On the first day of school, the students meet outside the building and agree on the following plan: The first student will enter the school and open all of the lockers. The second student will then enter the school and close every locker with an even number (2, 4, 6, 8, . . .). The third student will then "reverse" every third locker. That is, if the locker is closed, he will open it; if the locker is open, he will close it. The fourth student will then reverse every fourth locker, and so on until all 1,000 students in turn have entered the building and reversed the proper lockers. Which lockers will finally remain open?

Discussion It seems rather futile to attempt this experiment with 1,000 lockers, so let's take a look at 20 lockers and 20 students and try to find a pattern:



Locker #	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Student 1	O	O	O	O	O	O	O	O	O	O	O	O	O	O	O	O	O	O	O	O
2		C	O	C	O	C	O	C	O	C	O	C	O	C	O	C	O	C	O	C
3			C	C	O	O	O	C	C	C	O	O	O	C	C	C	O	O	O	C
4				O	O	O	O	O	C	C	O	C	O	C	C	O	O	O	O	O
5					C	O	O	O	C	O	O	C	O	C	O	O	O	O	O	C
6						C	O	O	C	O	O	O	O	C	O	O	O	C	O	C
7							C	O	C	O	O	O	O	O	O	O	O	C	O	C
8								C	C	O	O	O	O	O	O	C	O	C	O	C
9									O	O	O	O	O	O	O	C	O	O	O	C
10										C	O	O	O	O	O	C	O	O	O	O
11											C	O	O	O	O	C	O	O	O	O
12												C	O	O	O	C	O	O	O	O
13													C	O	O	C	O	O	O	O
14														C	O	C	O	O	O	O
15															C	C	O	O	O	O
16																O	O	O	O	O
17																	C	O	O	O
18																		C	O	O
19																			C	O
20																				C

Figure 1-2

In our smaller illustration in Figure 1-2, the lockers with numbers 1, 4, 9, and 16 remain open while all others are closed. Thus we conclude that those lockers with numbers that are perfect squares will remain open when the process has been completed by all 1,000 students. Notice that a locker "change" corresponds to a divisor of the locker number. An odd number of "changes" is required to leave a locker open. Which kinds of numbers have an odd number of divisors? Only the perfect squares!

This problem has embedded in it several basic mathematical concepts, namely factors, divisors, and the number-theoretic properties of composites and perfect squares.

PROBLEM The Greens are having a party. The first time the doorbell rings, 1 guest enters; on the second ring, 3 guests enter; on the third ring, 5 guests enter, and so on. That is, on each successive ring, the entering group is 2 guests larger than the preceding group. How many guests will enter on the fifteenth ring? How many guests will be present after the fifteenth ring?

Discussion Again, let's make a table and search for a pattern.

Ring number	*People enter*	*Total present*
1	1	1
2	3	4
3	5	9
4	7	16
5	9	25
6	11	36
.	.	.
.	.	.
.	.	.
n	$(2n - 1)$	n^2

This means that on the fifteenth ring, $(2 \cdot 15 - 1)$ or 29 people will enter; a total of 225 people will have arrived after the fifteenth ring.

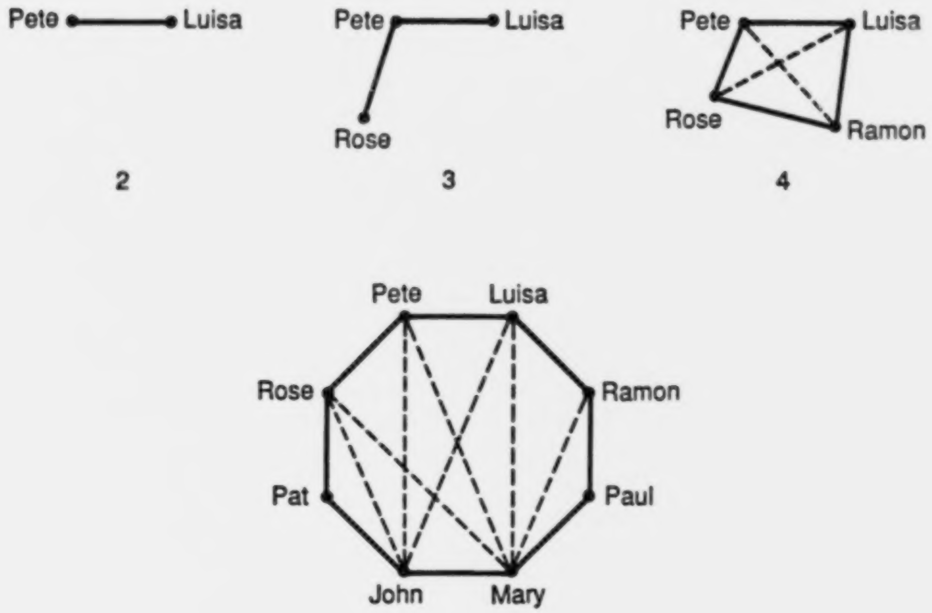
Notice that this problem illustrates the mathematical fact that the sum of the first n odd integers is n^2.

2. The solution of the problem leads to a generalization.

Some problems lend themselves beautifully to a generalization. That is, the solution reveals a pattern that can be expressed algebraically to encompass a variety of similar cases. In fact, a major task for mathematicians is to develop these generalizations. This is a major power of algebra—the means by which we represent a generalization.

PROBLEM There are 8 people in a room. Each person shakes hands with each of the other people once and only once. How many handshakes are there?

Discussion This problem may be attacked by considering a sequence of events involving an increasing number of persons. We will begin with two people and increase by one person at a time, as shown in Figure 1-3.



2 3 4

Figure 1-3

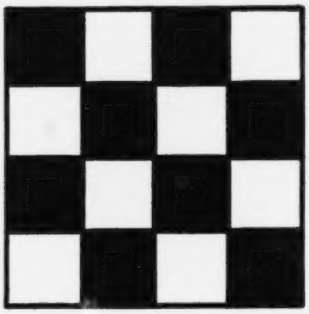
Observing the number pattern, the answer can be found to be 28 handshakes. Notice that this problem situation can be represented mathematically by a geometric model of a convex octagon, where the vertices represent the people and the edges and diagonals represent the handshakes. (Only a few representative handshakes are shown in Figure 1-3.) Thus we are asking students to identify the number of diagonals plus the number of sides of a convex polygon. The generalization of this problem can be found by applying the method of finite differences. This yields the formula:

$$H = \frac{n(n-3)}{2} + n$$

which reduces to

$$H = \frac{n(n-1)}{2}$$

Thus, a generalization has been found that resolves all problems that can be represented by this geometric model.



A 4 x 4 checkerboard

Figure 1-4

PROBLEM How many squares are there on a 4 × 4 checkerboard, as shown in Figure 1-4?

Discussion This problem may be approached by counting the number of squares of all sizes on a 1 × 1 checkerboard, then on a 2 × 2 checkerboard, and so on.

Number of squares

		1 × 1	2 × 2	3 × 3	4 × 4	Total
Board size	1 × 1	1				1
	2 × 2	4	1			5
	3 × 3	9	4	1		14
	4 × 4	16	9	4	1	30

There are 30 squares on this checkerboard.
We can now generalize this problem. The formula

$$S = \frac{n(n + 1)(2n + 1)}{6}$$

gives the number of squares of all sizes on any $n \times n$ checkerboard.

3. The problem is open-ended in that it affords an opportunity for extensions.

A problem is not necessarily finished when an answer has been found. The solution should suggest variations on parts of the original problem. The problem might be changed from a two-dimensional plane-geometry problem to a three-space situation. Circles become spheres; rectangles become "boxes." We should extend the problem by asking "What if . . ." questions. What if we hold one variable constant and let another change? What if the shapes of the given figure vary? What if the dimensions change? The value of any problem increases when it is extended.

PROBLEM Into how many non-overlapping linear regions do n different points on a line separate the line?

Discussion The problem can be extended to two dimensions: Into how many non-overlapping plane regions do n lines, no three of which are concurrent and none of which are parallel, separate the plane?
The same problem can then be extended to 3 dimensions: Into how many non-overlapping space regions do n planes that do not intersect in a line and that are not parallel separate the space?

PROBLEM For which positive integers a, b, and c will $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$?

Discussion An immediate solution is to let a, b, and c each equal 3. This yields the solution $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$. There is nothing in the statement of the original problem to preclude this solution, and it does satisfy the problem. However, it clearly leaves us looking for more. A natural extension, then, is to add the condition that $a \neq b \neq c$. Now a solution is $a = 2$, $b = 3$, $c = 6$, since $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$.
A further extension might be to ask, For which positive integers, n, can we express n as the sum of the reciprocals of some number of different positive integers?

PROBLEM A party of 18 people went to a restaurant for dinner. The restaurant has tables that seat 4 or 6 people. Show how the maitre d' can seat the party.

Discussion They can be seated in two different ways:

(a) 3 tables of 6
(b) 1 table of 6 and 3 tables of 4

The problem can now be extended: What if two more people join the group? Now, how might they be seated? The answer is still only two ways, but they are different:

(a) 5 tables of 4
(b) 2 tables of 6 and 2 tables of 4.

What if 1 table that seats 8 people were available? What if the party consisted of 21 people?

4. The problem lends itself to a variety of solutions.

A problem can often be solved in many different ways. It is of more value to the problem-solving process to solve one problem in four ways than to solve four problems, each in one way.

PROBLEM Prove: If a point lies on one side of a triangle and is equally distant from the three vertices, then the triangle is a right triangle.

Discussion 1 Let P be the point on side AB of triangle ABC such that $AP = BP = CP$. Then label the angle measures x and y, using the properties of the isosceles triangles APC and BPC. (See Figure 1-5.)

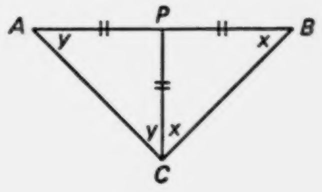


Figure 1-5

Now, since the sum of the measures of the angles of the triangle ABC equals 180°

$$\begin{aligned} 2x + 2y &= 180 \\ x + y &= 90 \end{aligned}$$

which is the required proof.

Discussion 2 Triangle ABC can be inscribed in a circle, as shown in Figure 1-6. Since point P is equidistant from A, B, and C, then P must be the center of the circle.

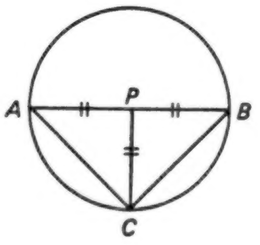


Figure 1-6

This makes *APB* a diameter of the circle, and an angle inscribed in a semicircle is a right angle.

Discussion 3 Extend segment *CP* its own length along ray *CP* to point *D*. Draw *DA* and *DB*. (See Figure 1-7.)

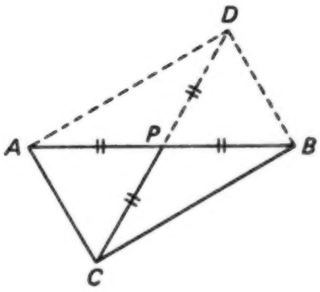


Figure 1-7

Now consider quadrilateral *CADB*. Since its diagonals bisect each other (*CP* = *PD* by construction, *AP* = *PB* by the given conditions), the quadrilateral must be a parallelogram. Furthermore, since the diagonals are congruent (*APB* = *DPC*), the parallelogram must be a rectangle. Hence angle *C* is a right angle.

PROBLEM A farmer has some pigs and some chickens. He finds that together they have 70 heads and 200 legs. How many pigs and how many chickens does he have?

Discussion 1 The problem can be solved algebraically by using two equations in two variables:

$$\begin{aligned} 1p + 1c &= 70 \quad \text{(heads)} \\ 4p + 2c &= 200 \quad \text{(legs)} \\ p &= 30 \\ c &= 40 \end{aligned}$$

The farmer has 30 pigs and 40 chickens.

Discussion 2 A series of successive approximations will also enable students to find the answer:

CHICKENS		PIGS		TOTAL		
Number of heads	Number of legs	Number of heads	Number of legs	Number of heads	Number of legs	
70	140	0	0	70	140	(Not enough legs)
50	100	20	80	70	180	(Still not enough legs)
40	80	30	120	70	200	

Figure 1-8

Discussion 3 Use the idea of a one-to-one correspondence. All chickens stand on one leg, all pigs stand on their hind legs. Thus, the farmer will see 70 heads and 100 legs touching the ground. The extra legs must belong to the pigs, since the chickens have one leg per head touching the ground. Thus there are 30 pigs and 40 chickens.

PROBLEM Find all rectangles with integral sides whose area and perimeter are numerically equal.

Discussion 1 Using the diagram in Figure 1-9, we express the problem algebraically:

$$2a + 2b = ab$$

or

$$\frac{ab}{a + b} = 2$$

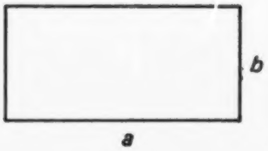


Figure 1-9

A first solution is the 4 × 4 rectangle (square) whose perimeter and area are both numerically 16. However, another set of values that satisfies the equation is $a = 6$ and $b = 3$. This rectangle has a perimeter and an area numerically equal to 18. These are the only two answers to this problem, unless we consider the degenerate case, namely the 0 × 0 rectangle.

Discussion 2 We prepare a series of tables of all rectangles with integral sides, together with their perimeter and area.

Case I

L	*W*	*P*	*A*
1	1	4	1
1	2	6	2
1	3	8	3
1	4	10	4

Since the perimeter is increasing more rapidly than the area, we need go no further.

Case II

L	*W*	*P*	*A*
2	2	8	4
2	3	10	6
2	4	12	8
2	5	14	10

Since the perimeter and the area are increasing at the same rate, we need go no further.

Case III

L	*W*	*P*	*A*
3	3	12	9
3	4	14	12
3	5	16	15
3	6	18	18

(which is an answer)

The perimeter and the area both = 18. Thus, the length is 3 and the width is 6.

Case IV

L	*W*	*P*	*A*
4	4	16	16

The perimeter and the area both = 16. Thus, the length is 4 and the width is 4.

Case V

L	*W*	*P*	*A*
5	5	20	25
5	6	22	30
5	7	24	35

Since the area is increasing more rapidly than the perimeter, we need go no further.
No further tables are needed.

5. The problem should be interesting and challenging to the students.

In order to be considered a problem, the confrontation must be recognized as such by the student. In other words, he or she must *want* to solve it. Thus, the setting should be one to which the student can relate or which appeals to his or her imagination. Topical areas such as science fiction, sports, and music are among those that have been shown to have appeal for many students. Teachers should become familiar with the lifestyles of their students in order to choose appropriate settings for problems.

In summary, a good problem will have several of the characteristics we have just discussed. Although we have used different problems to illustrate each specific characteristic, every illustration contained more than one of these desirable properties. For example, the locker problem, which we used to illustrate mathematical content, also affords an opportunity for multiple solutions (the problem can be "acted out" or simulated using students or bottle caps to represent the lockers), it is interesting and challenging, and, for more advanced students, the problem can be extended into some of the elementary properties of number theory.

WHAT MAKES A GOOD TEACHER OF PROBLEM SOLVING?

As is true in all education, the teacher is the crucial catalyst in the classroom. If we are to create good problem solvers among our students, the teacher's role is paramount. Without an interested, energetic, enthusiastic, knowledgeable, and involved guide, nothing positive will take place.

If the students are to succeed in learning problem solving, the teacher must have a positive attitude towards the problem-solving process itself. This means that teachers must prepare carefully for problem solving and be aware of the opportunities for problem solving that present themselves in everyday classroom situations. You may have to modify a problem to assure its pedagogical value—its scope may have to be reduced, or the problem restated in terms of the students' experiences. Knowing your students helps you make these choices. Problems should be solved in class carefully; the teacher should allow for and encourage a wide variety of approaches, ideas, questions, solutions, and discussions. Teachers must exhibit restraint and not be too quick to "give" the "correct solution." Teachers must be confident in class and must exhibit the same enthusiasm for the problem-solving process that they wish to instill in their students.

Some teachers dislike problem solving because they have not had enough successful experiences in this area. Practice will provide these experiences. Teachers who encourage their students to solve problems, who make their students think, and who ask carefully worded questions (rather than merely giving answers), will provide their students with a rich problem-solving experience.

CHAPTER TWO

A Workable Set of Heuristics

WHAT ARE HEURISTICS?

Problem solving is a *process*. The process starts when the initial encounter with the problem is made and ends when the obtained answer is reviewed in light of the given information. Children must learn this process if they are to deal successfully with the problems they will meet in school and elsewhere. This process is complex and difficult to learn. It consists of a series of tasks and thought processes that are loosely linked together to form what is called a *set of heuristics* or a *heuristic pattern*. They are a set of suggestions and questions that a person must go through in order to resolve a dilemma.

Heuristics should not be confused with algorithms. Algorithms are schemas that are applied to single-class problems. In computer language, they are programs that can be called up to solve the specific problems or classes of problems for which they were developed. For each problem or class of problems, there is a specific algorithm. If one chooses and properly applies the appropriate algorithm, and makes no arithmetic or mechanical errors, then the answer that is obtained will be correct. In contrast, heuristics are general and are applicable to all classes of problems. They provide the direction needed by all people to approach, understand, and obtain answers to problems that confront them.

There is no single set of heuristics for problem solving, although several people have put forth workable models. Whether the student follows the one put forth by Polya or the one that appears in this chapter is not important; what is important is that students learn some set of carefully developed heuristics, and that they develop the habit of applying these heuristics in all problem-solving situations.

It is apparent that simply providing students with a set of heuristics to follow would be of little value. There is quite a difference between understanding a strategy on an intellectual plane (recognizing and describing it) and being able to apply the strategy. Thus, we must do more than merely hand the heuristics to the student; rather, instruction must focus on the thinking that the problem solver goes through as he or she considers a problem. It is the process—not the answer—that is problem solving.

Applying heuristics is a difficult skill in itself. We must spend time showing students how (and when) to use each of the heuristics—a prescriptive approach rather than a merely descriptive one. Then we must constantly use and refer to the heuristics as our students solve problems.

A SET OF HEURISTICS TO USE

Over the years, several sets of heuristics have been developed to assist students in problem solving. In the main, they are quite similar. Figure 2-1 presents a flowchart of a set of heuristics that has proven to be successful with students and teachers at all levels of instruction. This heuristic plan represents a continuum of thought that every person should use when confronted by a problem-solving situation. Since it is a continuum, its parts are not discrete. In fact, "Read the problem" and "Explore" could easily be considered at the same time under a single heading such as "Think." As the problem solver is exploring, he or she is considering what strategy to select.

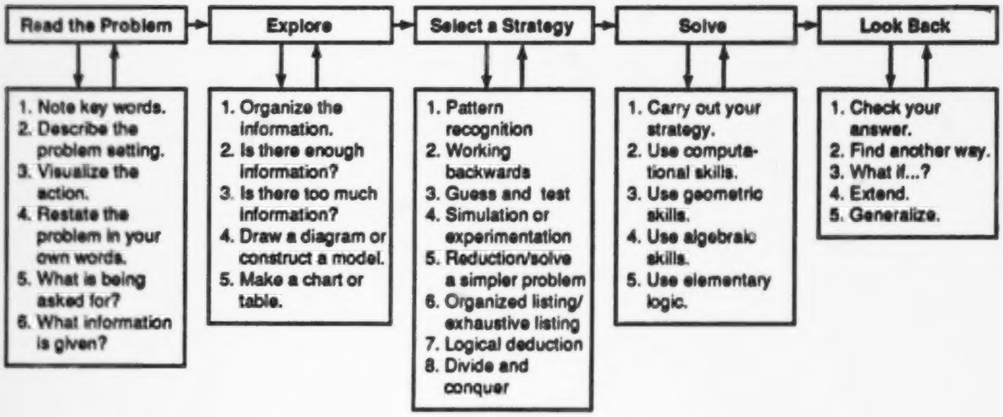

Read the Problem
1. Note key words.
2. Describe the problem setting.
3. Visualize the action.
4. Restate the problem in your own words.
5. What is being asked for?
6. What information is given?
Explore
1. Organize the information.
2. Is there enough information?
3. Is there too much information?
4. Draw a diagram or construct a model.
5. Make a chart or table.
Select a Strategy
1. Pattern recognition
2. Working backwards
3. Guess and test
4. Simulation or experimentation
5. Reduction/solve a simpler problem
6. Organized listing/ exhaustive listing
7. Logical deduction
8. Divide and conquer
Solve
1. Carry out your strategy.
2. Use computational skills.
3. Use geometric skills.
4. Use algebraic skills.
5. Use elementary logic.
Look Back
1. Check your answer.
2. Find another way.
3. What if...?
4. Extend.
5. Generalize.

Figure 2-1

1. Read the problem.

1. Note the key words.
2. Describe the problem setting.
3. Visualize the action.
4. Restate the problem in your own words.
5. What is being asked for?
6. What information is given?

PROBLEM What are the prime factors of 36?

Discussion Notice that the key word is "prime." A student could list all of the factors of 36, but only 2 and 3 are prime factors.

PROBLEM Jeff weighs 180 pounds. His sister, Nancy, weighs 118 pounds. Scott weighs 46 pounds more than Nancy. What is the average weight of the three people?

Discussion Here the key words are "more than" and "average." Words such as "more than," "less than," and "subtracted from" are often overlooked by students.

PROBLEM Lucy leaves school for her home at 3:00. Thirty minutes later, her sister, Carla, leaves school and overtakes Lucy at 4:00. If Carla travels at 10 miles per hour on her bike, how far from school does she catch up to Lucy?

Discussion Can you visualize the action? Can you describe what is taking place? The key word here is "overtake."

2. Explore.

1. Organize the information.
2. Is there enough information?
3. Is there too much information?
4. Draw a diagram or construct a model.
5. Make a chart or a table.

PROBLEM Two ships leave the same port at 9:00 A.M. The first ship travels on a course whose bearing is 20° at a rate of 5 knots. The second ship takes a course whose bearing is 110° and travels at a speed of 12 knots. How far apart are the ships after 2 hours of traveling?

Discussion Students must draw a diagram (such as Figure 2-2) in order to organize the given information. There are several key words and ideas in the problem: "course," "bearing," "knots." Notice that the 9:00 A.M. departure time is excess information.

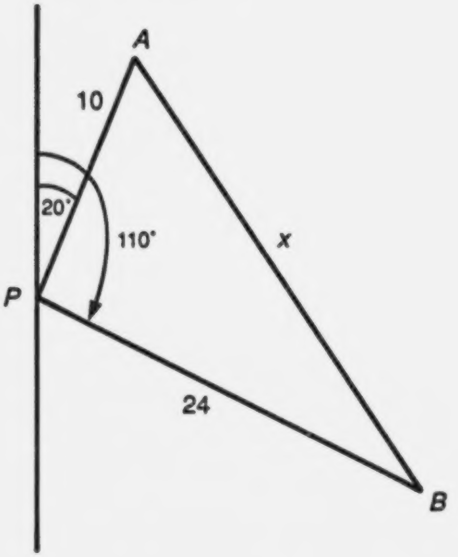


Figure 2-2

The difference between a 20° bearing and a 110° bearing makes angle *APB* a right angle. The Pythagorean relationship gives the answer: 26 nautical miles.

PROBLEM Four people work for the Keystone Crystal Works. Mr. Jones earns \$36,000 per year, Mr. Snidely earns \$28,000 per year, and Miss Magolin earns \$32,000 per year. Ms. Batcher's salary differs from the mean salary of the other three people by \$5,000. What is Ms. Batcher's salary?

Discussion The key words here are "differs" and "mean." After organizing the information, the students should calculate the mean salary for the first 3 people as \$32,000. However, there is not enough information to provide a unique answer, since both \$27,000 and \$37,000 differ from the mean by \$5,000.

PROBLEM A log is cut into 4 pieces in 12 seconds. At the same rate, how many seconds will it take to cut the log into 5 pieces?

Discussion Notice the key phrase "at the same rate." In many cases, students will react to this problem as an exercise in proportion:

$$\frac{4}{12} = \frac{5}{x}$$

They should draw a diagram, as shown in Figure 2-3. They will quickly see that the solution process should focus on the number of cuts needed to get the required number of pieces, rather than on the actual number of pieces.

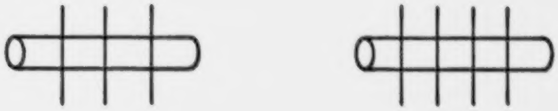


Figure 2-3

Once the diagram has been interpreted, students should realize that they do not need proportions. They see that 3 cuts produced 4 pieces, and thus they need 4 cuts to produce 5 pieces. Since 3 cuts were made in 12 seconds, each cut requires 4 seconds. Therefore, the necessary 4 cuts will require 16 seconds.

PROBLEM A farmer wishes to purchase a piece of land that is adjacent to his farm. The real estate agent tells him that the plot is triangular in shape, with sides of 20, 75, and 45 meters. This land will cost only \$5.58 a square meter. How much should the farmer pay for the piece of land?

Discussion The problem can be solved in a geometry class by using Heron's formula:

$$A = \sqrt{s(s - a)(s - b)(s - c)}$$

(where s = semiperimeter). When the student draws an appropriate diagram (as shown in Figure 2-4), it becomes readily apparent that there is no triangle, since the sum of two of the sides is not greater than the third side (45 + 20 < 75).

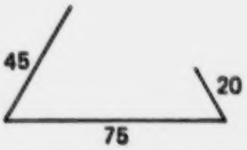


Figure 2-4

PROBLEM Claire drove from Philadelphia to Brooklyn, then to Manhattan, then to Scarsdale. She returned over the same route. From Philadelphia to Brooklyn is 93 miles. From Brooklyn to Manhattan is 11 miles. The trip from Scarsdale to Philadelphia was 131 miles. What is the distance between Manhattan and Scarsdale?

Discussion While the problem sounds cumbersome, confusing, and full of excess data, it is simplified by the use of a diagram, as in Figure 2-5:

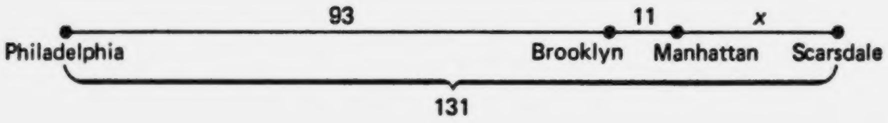


Figure 2-5

Philadelphia to Brooklyn	=	93
Brooklyn to Manhattan	=	11
Manhattan to Scarsdale	=	x
131	=	$104 + x$

PROBLEM The Sandwich Shop serves pizza at 95¢ a slice, ice cream cones for 85¢, and soft drinks for 60¢. Jesse bought a slice of pizza and a soft drink. How much change did he receive?

Discussion This problem contains excess information and, at the same time, has insufficient data to solve it. The cost of the ice cream is excess, and the answer cannot be found because the amount of money given to the cashier is not known.

PROBLEM Find the next three numbers in the sequence 2, 3, 5, 8, 12.

Discussion We record the numbers in a row as in Figure 2-6 and examine the differences between successive terms.

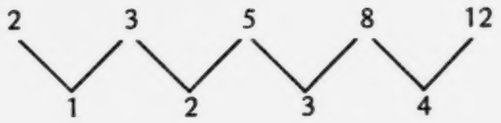


Figure 2-6

The first differences are the natural numbers, and we can find the next three terms (17, 23, 30) quite easily.

PROBLEM Into how many unique segments do *n* points divide a given line segment?

Discussion First draw a diagram as in Figure 2-7.

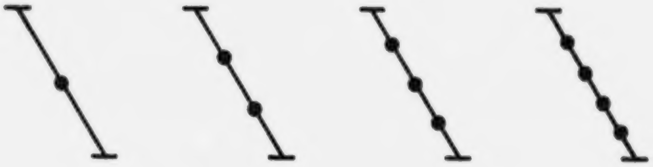


Figure 2-7

Then organize the data by completing the table. Look for a pattern.

Number of points	1	2	3	4	. . .	*n*
Number of segments	2	3	4	5		$(n + 1)$

3. Select a strategy.

1. Pattern recognition
2. Working Backwards
3. Guess and Test
4. Simulation or experimentation
5. Reduction/solve a simpler problem
6. Organized listing/exhaustive listing
7. Logical deduction
8. Divide and conquer

Rarely does one single strategy suffice to solve a problem. Rather, it is usually a combination of strategies that is required. For instance, when using the Guess and Test strategy, a table should be made to keep track of the guesses and their corresponding results. Similarly, organizing the data in tabular form often reveals a pattern. In our discussion of the problems that follow, we will identify various strategies used to achieve the answer. Many of these problems will have solutions other than the ones we have chosen to present. Have your students try to find alternate solutions, since a variety of solutions to a single problem are more revealing than single solutions to each of many problems.

PROBLEM What is the units digit of 234^{17}?

Discussion The value of 243^{17} is virtually impossible to arrive at without the use of technology. However, the final digit can be determined by looking for a pattern. Examine the first 8 powers of 3:

$$\begin{array}{llll} 3^0 = 1 & & 3^4 = & 81 \\ 3^1 = 3 & & 3^5 = & 243 \\ 3^2 = 9 & & 3^6 = & 729 \\ 3^3 = 27 & & 3^7 = & 2187 \end{array}$$

Notice the repetitive pattern of the units digit, namely 1, 3, 9, 7, 1, 3, 9, 7, ... This is a mod 4 system. Thus, $17 = 1 \bmod 4$. Therefore the final digit of $243^{17} = 3$.

PROBLEM Alex, Patti, and Charlotte decide to play a game of cards and agree to the following procedure. When a player loses a game, she will double the amount of money that each of the other players already has. First Alex loses a hand and doubles the amounts of money that Patti and Charlotte have. Then Patti loses a hand and doubles the amounts of money that Charlotte and Alex each have. Then Charlotte loses a hand and doubles the amounts of money that Alex and Patti each have. The three players then decide to quit and find that each now has $8. How much money did Patti start with?

Discussion This problem can be solved in an algebra class by writing a system of three equations in three variables. Once this is done, the system can be solved easily. However, arriving at the system of equations is quite difficult and requires a complicated table.

An alternative approach would be to have the students solve the problem by working backwards. This quickly yields the solution at an artistic, less mathematically sophisticated level. At the same time, it provides the students with an additional strategy of problem solving: Working Backwards.

PROBLEM A piece of wire 52 inches long is cut into two segments of integral lengths. Each segment is then bent to form a square. The sum of the areas of the two squares is 97 square inches. Find the length of each segment of wire.

Discussion While this problem can again be solved very nicely by the use of algebra (a system of two equations in two variables), we will show the solution based on Guess and Test. We are looking for two perfect squares whose sum is 97. We prepare a table of our guesses:

$1^2 = 1 + 96 = 97$ (no)
$2^2 = 4 + 93 = 97$ (no)
$3^2 = 9 + 88 = 97$ (no)
$4^2 = 16 + 81 = 97$ (yes)

Thus 4 and 9 are the sides of the squares. Now we formulate the perimeters: 16 and 36 are the lengths of the two segments of wire. (See pages 39–40 for further discussion.)

PROBLEM A bridge that spans a bay is 1 mile long and is suspended from two supports, one at each end. As a result, when it expands a total of 2 feet from the summer heat, it "buckles" in the center, causing a bulge. How high is the bulge?

Discussion We will attack this problem by means of a simulation. The power of mathematics lies in its ability to simulate action—in this case, with a drawing.

Draw a diagram of the bridge, both before and after the expansion has caused it to buckle. (See Figure 2-8.)

We can now approximate the situation, using the diagram with two right triangles as shown in Figure 2-9:

The Pythagorean Theorem can now be applied to the right triangle:

$$\begin{aligned} x^2 + (2{,}640)^2 &= (2{,}641)^2 \\ x^2 + 6{,}969{,}600 &= 6{,}974{,}881 \\ x^2 &= 5{,}281 \\ x &= 72.67 \end{aligned}$$

Thus, the bulge is approximately 73 feet high.

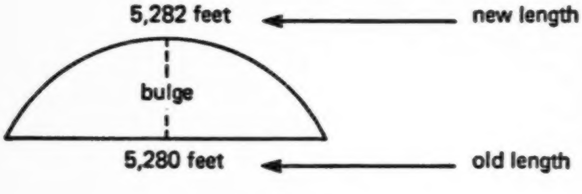


Figure 2-8

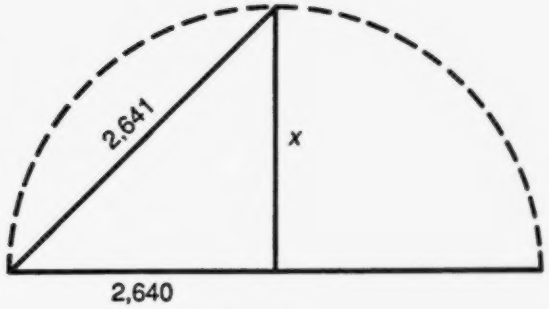


Figure 2-9

PROBLEM A flat display of baseballs in the form of a triangle is being prepared for the window of a local sporting goods store. There is 1 baseball in the top row; there are 2 baseballs in the second row; 3 in the third row; and so on, with each new row containing one more baseball than the row above it. The display contains 12 rows. (a) How many baseballs will be in the twelfth row? (b) How many baseballs will there be in the entire display?

Discussion A reasonable way of attacking this problem is to reduce the complexity of the problem. Let's start with 1 row, then 2 rows, then 3 rows, and so on, keeping a record as we go. Perhaps a pattern will emerge.

Row number	*Number of balls*	*Total number of balls*
1	1	1
2	2	3
3	3	6
4	4	10
5	5	15
.	.	.
.	.	.
.	.	.

The table quickly reveals the answer to part (a): There will be 12 baseballs in row 12. The third column shows that the total number of baseballs in the display is represented by the sum of the numbers of the rows to that point (the sum of the first n integers). Thus, if there are 12 rows, the answer will be $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 = 78$. Notice that this can be generalized by using the formula

$$S = \frac{n(n+1)}{2}$$

PROBLEM During the recent census, a man told the census-taker that he had three children. When asked their ages, he replied, "The product of their ages is 72. The sum of their ages is the same as my house number." The census-taker ran to the door and looked at the house number. "I still can't tell the ages of your children," she complained. What is the house number?

Discussion Prepare an exhaustive list of all whole number triples whose product is 72, together with their sums:

1-1-72 = 74
1-2-36 = 39
1-3-24 = 28
1-4-18 = 23
1-6-12 = 19
1-8- 9 = 18
2-2-18 = 22
2-3-12 = 17
2-4- 9 = 15
2-6- 6 = 14
3-3-8 = 14
3-4-6 = 13

Since there was still a question after seeing the sum of the ages (the house number), there had to be more than one set of factors whose sum equaled this number (3-3-8 and 2-6-6 both sum to 14). Thus the house number must have been 14.

PROBLEM Four married couples went to the baseball game last week. The wives' names are Carol, Sue, Jeanette, and Arlene. The husbands' names are Dan, Bob, Gary, and Frank. Bob and Jeanette are brother and sister. Jeanette and Frank were once engaged, but broke up when Jeanette met her husband. Arlene has a brother and a sister, but her husband is an only child. Carol is married to Gary. Who is married to whom?

Discussion Prepare a 4 × 4 matrix like the one shown below. Use the clues to eliminate the incorrect choices.

	Carol	Sue	Jeanette	Arlene
Frank	X	X	X	YES
Gary	YES	X	X	X
Bob	X	YES	X	X
Dan	X	X	YES	X

PROBLEM A sailboat is traveling along a course that is in the shape of an equilateral triangle. The first leg is 10 miles long, and the boat travels at a rate of 6 miles per hour. It covers the second leg at 5 miles per hour, and the third leg at 4 miles per hour. How long does the boat take to complete the course?

Discussion Since the course is in the shape of an equilateral triangle, each leg must be 10. Some students may "average" the rates as $\frac{6 + 5 + 4}{3}$ = 5 miles per hour. Dividing the 30-mile distance by 5, they will arrive at a time of 6 hours. This is incorrect! Each leg must have its time calculated separately; thus, divide and conquer.

Leg 1: 10 ÷ 5 = 2.00
Leg 2: 10 ÷ 6 = 1.67
Leg 3: 10 ÷ 4 = 2.50
Total time = 6.17 hours

PROBLEM Number the eight vertices of a cube with the numbers 1 through 8, so that the sum of the four numbers on the vertices of any face will be 18.

Discussion This problem is solved by using the Guess and Test strategy. Notice, however, that students should recognize that 8 + 1 = 9, 7 + 2 = 9, 6 + 3 = 9, 5 + 4 = 9. One possible solution is shown in Figure 2-10.

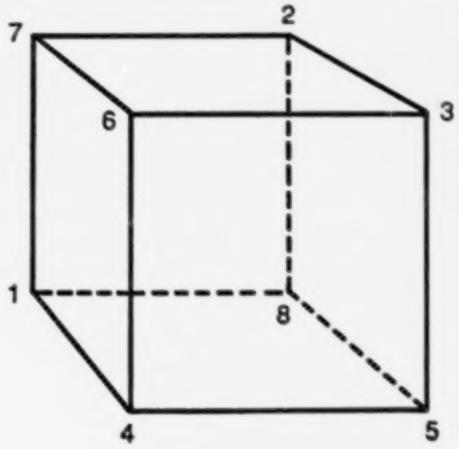


Figure 2-10

PROBLEM There are three natural numbers that are less than 1,000 and are both perfect squares and perfect cubes. Find these three numbers.

Discussion In this problem, it is particularly important that the students understand the question and the given information. The key words, "natural numbers," exclude 0 as a solution. The students might even be able to guess the first two numbers that satisfy the given conditions, namely 1 and 64. They should then make an exhaustive list of all the numbers less than 1,000 that are perfect squares, and a similar list of those that are perfect cubes. They then examine both lists to find the numbers that appear on both. Preparing the lists provides an excellent opportunity to use the hand-held calculator. (The third number is 729.)

PROBLEM Two couples sitting on a park bench pose for a picture. If neither couple wishes to be separated, what is the number of different arrangements for seating the couples that can be used for the photograph?

Discussion Simulate the action with a drawing. Let the letters *A,B* represent the first couple and the letters *C,D* represent the second couple. Remember that *A,B* is different from *B,A*. There will be 8 different arrangements.

4. Solve.

1. Carry out your strategy.
2. Use computational skills.
3. Use geometric skills.
4. Use algebraic skills.
5. Use elementary logic.

Once the problem has been understood and a strategy selected, the student should perform the mathematics necessary to arrive at an answer. We will include only one illustration in this section, since this is the area of the heuristic process where most teachers already spend considerable time. Although only one example is included, we do not consider this part of the heuristic plan to be unimportant. Indeed, it is imperative that students be able to arrive at a correct answer. However, the "answer" is not the ultimate goal—the "solution" is. We maintain that the solution is the process by which the answer is obtained.

PROBLEM A basketball player can dribble a ball 18 times in 10 seconds; how many dribbles can she do in 1 minute? How long does it take to do 63 dribbles?

Discussion Students decide to construct a table as part of their solution to this problem. Some students may make a table that only carries the problem out through 30 seconds or so; others will carry the table all the way out to 60 seconds.

Seconds	10	20	30	40	50	60
Dribbles	18	36	54	72	90	108

It is interesting to note that the problem does not state that the rate stays constant. Some students may wish to discuss the fact that, as time passes, the ballplayer might become tired and do fewer dribbles in each time period.

Some simple interpolation is required to find the answer to the second part of the problem. Use a proportion.

5. Look back.

1. Check your answer.
2. Find another way.
3. What if . . . ?
4. Extend.
5. Generalize.

The answer is only a part of the solution. There is still more to be done once an answer has been arrived at. The purpose of having students solve problems is to make them better problem solvers—that is, to learn the process. This process does not end when an answer has been obtained. The fifth step of the heuristic plan, Look Back, is a vital part of the process. It is in this step that we verify the answer, review the computation, look for alternate solutions, and extend the problem to further the students' knowledge of mathematics and the way that mathematicians work.

PROBLEM What is the largest 3-digit number that is divisible by both 9 and 7?

Discussion Some students may quickly answer 63. When they look back, they should note that this does not satisfy all the conditions. Although 63 *is* divisible by both 9 and 7, it is not a 3-digit number.

Some students may give 315 as an answer. Again, looking back reveals that all of the conditions have not been met. This is not the *largest* 3-digit number that is divisible by both 9 and 7. The answer that meets all the given conditions is 945.

PROBLEM Sue and Rita collect coins. Sue decided to give up her collection. First, she gave half of them and half a coin more to Renee. Then she gave half of the coins that were left and half a coin more to Tanya. This left her with only 1 coin, which she gave to Rita. How many coins did Sue start with?

Discussion This problem can be solved by using algebra. However, the equation found is rather complicated:

$$x - \left(\frac{x}{2} + \frac{1}{2}\right) - \left[\frac{x - \left(\frac{x}{2} + \frac{1}{2}\right)}{2} + \frac{1}{2}\right] = 1$$

Once this equation has been formulated and solved, the students should be made aware that an alternate solution would be to work backwards. We can assume that Sue did not cut any coins in half.

Since 1 coin remained after Sue's last gift, she must have had 3 coins before giving the 1 coin to Rita (half of 3 is 1½, plus ½ equals 2). So her gift to Tanya was 2 coins. Working backwards in a similar manner, we find that Sue must have started with 7 coins and given (3½ + ½) or 4 coins to Renee.

PROBLEM A dartboard is to be designed in the shape of a rectangle, divided into four equal quadrants. You are to assign four integers from the set 1 to 15 to the quadrants. You will throw four darts; each lands on the board. The following conditions must be met:

(a) No integer may be used more than once.
(b) The maximum possible sum is 52.
(c) The minimum possible sum is 12.
(d) The sum must be even.

What numbers would you choose?

Discussion Let's look at the conditions, one at a time. If the maximum sum for the 4 darts is to be 52, then the number 13 must be the largest one selected. If the minimum sum is to be 12, then the smallest integer on the board must be 3. Thus

3 and 13 represent the smallest and largest numbers; the other two must be in between.

We extend the problem by examining the properties of addition for even and odd integers. All four numbers must be even or all four numbers must be odd in order to have an even sum, since the sum of any number of even numbers is always even, while only the sum of an even number of odd integers is even. The numbers 3 and 13 have been determined by conditions (b) and (c); thus, the remaining two numbers must also be odd. Students discover that the other two numbers can be chosen from 5, 7, 9, and 11."What if . . ." can also be employed with this problem. For example, suppose we add another condition:

What if the sum of 50 could *not* be obtained?

Notice that this technique maintains the same setting and the same question, but a condition has been changed. By logic and/or Guess and Test, this "What if . . ." eliminates 11 as a possible choice. We can play "What if . . ." again:

What if the sum of 14 could *not* be obtained?

These two added conditions make the answer to the problem unique (3, 7, 9, 13).

APPLYING THE HEURISTICS

Now that each step of the heuristic process has been presented, discussed, and illustrated, let's apply the model to several problems. As the solutions are developed, be certain that you are aware of the thought processes being utilized in each step. Remember, problem solving is a *process*; the answer is merely the final outcome.

PROBLEM A piece of wire 52 inches long is cut into two pieces, and squares are formed from each of the pieces. The sum of the areas of the two squares is 97 square inches, and the sides are integral lengths. What is the length of the side of each square?

Discussion

1. Read the problem.

Describe the setting. Visualize the action. Restate the problem. What is being asked? What information is given? What are the key facts?

The key facts in the problem are "52 inches long," "two squares," "the sum of the areas is 97 square inches," and "the lengths of the sides are integral." We are asked to find the length of a side of each square.

2. Explore.

Make a drawing.

Let the drawing show two squares. (See Figure 2-11.) Use x and y to represent the lengths of the sides of the two squares. Remember that a square has four equal sides.

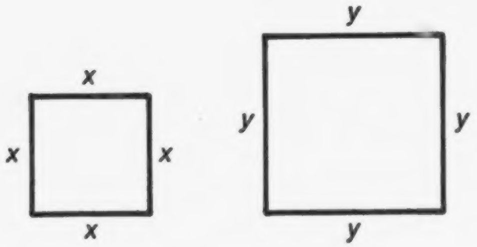


Figure 2-11

3. Select a strategy.

Use algebra.

$$\begin{aligned} x^2 + y^2 &= 97 \\ 4x + 4y &= 52 \end{aligned}$$

4. Solve.

Carry through your strategy.

Solve the two equations simultaneously.

$$\begin{aligned} x^2 + y^2 &= 97 \\ 4x + 4y &= 52 \\ x + y &= 13 \\ x^2 + (13 - x)^2 &= 97 \end{aligned}$$

$$\begin{aligned} x^2 + 169 - 26x + x^2 &= 97 \\ 2x^2 - 26x + 72 &= 0 \\ x^2 - 13x + 36 &= 0 \\ (x - 4)(x - 9) &= 0 \end{aligned}$$

$$\begin{array}{c|c} x = 4 & x = 9 \\ y = 9 & y = 4 \end{array}$$

The required lengths are 4 and 9.

5. Look back.

Check your answer. Find another way. Ask "What if . . . ?" Extend.

Try 4 and 9 as the sides of the two squares. Is the sum of the two areas 97? ($4^2 + 9^2 = 16 + 81 = 97$.) Yes. Is the sum of the perimeters 52? ($4 \times 4 + 4 \times 9 = 16 + 36 = 52$.) Yes. The answers satisfy.
Can you find another way? Let's guess and test. Since the sum of the two areas is 97, and the sides are integral, which two perfect squares add up to 97?

1 + 96	(96 is not a perfect square.)
4 + 93	(93 is not a perfect square.)
9 + 88	(88 is not a perfect square.)
16 + 81	These work! The answers are 4 and 9.

What if the two squares had been formed from a length of wire that was 104 inches and the sum of their areas was 388 square inches? Notice that the same process can be used, although the answers are different. Extend the problem by examining the relationships between the ratios of the lengths of the sides and the ratios of the areas. Note that the ratio of the areas is the square of the ratio of the lengths (4:9 and 16:81).

PROBLEM The train trip between Boston and New York takes exactly 5 hours. A train leaves Boston for New York every hour on the hour, while a train leaves New York for Boston every hour on the half-hour. A student at Harvard takes the New York-bound train from Boston at 10:00 A.M. How many Boston-bound trains will she pass before she arrives in New York?

Discussion

1. Read the problem.

Describe the setting. Visualize the action. Restate the problem. What is being asked? What information is given? What are the key facts?

The key facts are "the time of the trip is 5 hours," "from Boston to New York is every hour on the hour," "from New York to Boston is every hour on the half-hour," and "she departs from Boston at 10:00 A.M." The question is, how many Boston-bound trains did she pass?

2. Explore.

We want to find out which trains she will pass. The first one will be the one scheduled to arrive in Boston at 10:30 A.M. Since she arrives in New York at 3:00 P.M., the last train she will pass is the one that leaves New York at 2:30 P.M.

3. Select a strategy.

Make a list.

Trains passed	*Arrives Boston*	*Leaves New York*
First	10:30 A.M.	
.	.	.
.	.	.
.	.	.
Last		2:30 P.M.

4. Solve.

Carry out your strategy.

Complete the list.

Trains passed	*Arrives Boston*	*Leaves New York*
1	10:30 A.M.	5:30 A.M.
2	11:30 A.M.	6:30 A.M.
3	12:30 P.M.	7:30 A.M.
4	1:30 P.M.	8:30 A.M.
5	2:30 P.M.	9:30 A.M.
6	3:30 P.M.	10:30 A.M.
7	4:30 P.M.	11:30 A.M.
8	5:30 P.M.	12:30 P.M.
9	6:30 P.M.	1:30 P.M.
10	7:30 P.M.	2:30 P.M.

She will pass 10 trains.

5. Look back.

Find another way. Ask "What if . . . ?"

Another way to solve the problem is to draw a time line. The top line shows that she will see 5 trains that are already underway when she leaves at 10:00 A.M. The bottom line shows the track at 3:00 P.M., considering only New York-to-Boston trains.

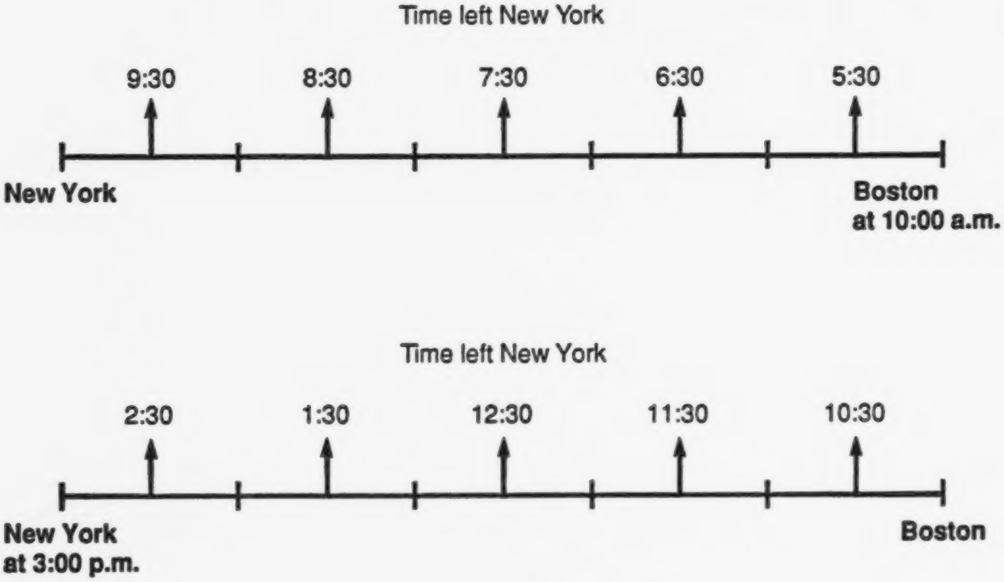
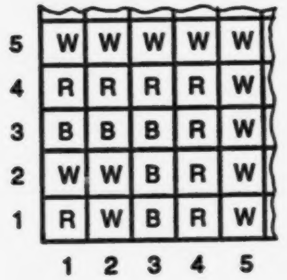

Time left New York
9:30
8:30
7:30
6:30
5:30
New York
Boston
at 10:00 a.m.
Time left New York
2:30
1:30
12:30
11:30
10:30
New York
at 3:00 p.m.
Boston

Figure 2-12

What if we want to know how many trains are on the tracks between New York and Boston during the 5-hour trip? (This includes trains going in both directions.)

PROBLEM A room that is 18 feet square is going to be tiled using tiles that are 1 foot square. The tiles come in three colors: red, white, and blue. The floor will be tiled in the manner shown in Figure 2-13.



5	W	W	W	W	W
4	R	R	R	R	W
3	B	B	B	R	W
2	W	W	B	R	W
1	R	W	B	R	W
	1	2	3	4	5

R = Red tiles
W = White tiles
B = Blue tiles

Figure 2-13

That is, the first tile is placed in a corner. Each consecutive layer forms a square about the previous layer (four such layers are shown in the figure). Layers 4, 5, and 6 will repeat the colors in layers 1, 2, and 3, and so on. This continues until the floor is completely covered. How many tiles of each color will be used?

Discussion 1. Read the problem.

Describe the setting. Visualize the action. Restate the problem. What is being asked? What information is given? What are the key facts?

The key facts are "room 18 feet square," "1-foot square tiles," "three colors," "layers to form squares," "colors alternate in rows and columns." The question is, how many tiles of each color will be used?

2. Explore.

Since the drawing is part of the problem, carry the partial drawing out further.

3. Select a strategy.

The drawing represents a simulation of the action.

4. Solve.

Carry through your strategy.

The most obvious way to arrive at the answer is to count the tiles. There are 96 red tiles, 108 white tiles, and 120 blue tiles (see Figure 2-14).

5. Look back.

18	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
17	W	W	W	W	W	W	W	W	W	W	W	W	W	W	W	W	W	B
16	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	W	B
15	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	R	W	B
14	W	W	W	W	W	W	W	W	W	W	W	W	W	W	B	R	W	B
13	R	R	R	R	R	R	R	R	R	R	R	R	R	W	B	R	W	B
12	B	B	B	B	B	B	B	B	B	B	B	B	R	W	B	R	W	B
11	W	W	W	W	W	W	W	W	W	W	W	B	R	W	B	R	W	B
10	R	R	R	R	R	R	R	R	R	R	W	B	R	W	B	R	W	B
9	B	B	B	B	B	B	B	B	B	R	W	B	R	W	B	R	W	B
8	W	W	W	W	W	W	W	W	B	R	W	B	R	W	B	R	W	B
7	R	R	R	R	R	R	R	W	B	R	W	B	R	W	B	R	W	B
6	B	B	B	B	B	B	R	W	B	R	W	B	R	W	B	R	W	B
5	W	W	W	W	W	B	R	W	B	R	W	B	R	W	B	R	W	B
4	R	R	R	R	W	B	R	W	B	R	W	B	R	W	B	R	W	B
3	B	B	B	R	W	B	R	W	B	R	W	B	R	W	B	R	W	B
2	W	W	B	R	W	B	R	W	B	R	W	B	R	W	B	R	W	B
1	R	W	B	R	W	B	R	W	B	R	W	B	R	W	B	R	W	B
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18

R = Red tiles
W = White tiles
B = Blue tiles

Figure 2-14

Check your answer. Find another way. Ask "What if . . . ?" Extend.

An 18-foot-square room must have 324 1-foot-square tiles to completely cover the floor. Add 96 + 108 + 120. Do you get 324? Yes. Does your finished drawing follow the pattern that was begun? Yes. The answer checks. Another way to solve this problem is to make a table.

Red		**White**		**Blue**	
Layer #	Number of tiles	Layer #	Number of tiles	Layer #	Number of tiles
1	1	2	3	3	5
4	7	5	9	6	11
7	13	8	15	9	17
10	.	11	.	12	.
13	.	14	.	15	.
16	.	17	.	18	.

The table reveals that, within a color, the number of tiles in any layer increases by 6. Thus, we may complete the table and add. Examining the table further reveals that the number of tiles in each layer is given by the formula $N_t = (2n - 1)$. Together with the formula for the sum of an arithmetic progression, we can find the total number of tiles for each color. In fact, we can use the formula

$$S = \frac{n}{2}[2a + (n - 1)d]$$

and find the sum of the tiles in each color without completing the table.

What if red tiles cost $1.00 each, white tiles cost 80¢ each, and blue tiles cost $1.20 each? How much would it cost to tile the floor?

What if the design were the same, but the colors could be interchanged? What would be the cheapest way to tile the floor?

CHAPTER THREE

The Pedagogy of Problem Solving

Problem solving is a process, so we must develop a set of heuristics to follow and then be certain to use it. Whether we use the heuristics developed in Chapter 2, the four-step heuristics of Polya, or some other set of seven, eight, or even more steps is not important. What is important is that students learn a heuristic model, develop an organized set of "questions" to ask themselves, and then constantly refer to them when confronted by a problem situation.

What can the teacher do to assist students in achieving this important goal? In this chapter, we will present methods that the teacher can employ in the classroom to assist students in the utilization of heuristics in their development of problem-solving skills. Activities are included for use with students.

1. Create a problem-solving environment in the classroom.

Problem solving is not only a process, but also a way of teaching. An attitude of inquiry should pervade all activities in the classroom. A risk-free environment must always be maintained; students should feel perfectly free to ask a question, think about a problem, or give any response that seems appropriate to them, without fear of criticism. These responses should all be discussed by the class, with the teacher serving as a moderator and resource person.

In this environment, every child should attain some measure of success. The old adage, "Nothing succeeds like success," holds true in the mathematics classroom. Remember that if a student refuses even to attempt to solve a problem, there can be no problem-solving activity. Choose the problems carefully. Begin with relatively simple problems so as to ensure a reasonable degree of success. If students are successful, they are likely to be "turned on" to problem solving, whereas repeated failure or constant frustration can have a devastating effect on motivation, attitude, and the desire to continue. However, remember that success must be truly earned, not just given. It means more than just a correct answer. Becoming absorbed in a problem and making a sustained attempt at solution is also success.

2. Encourage your students to solve problems.

For students to become good problem solvers, they must be constantly exposed to and involved in problem solving. In learning to swim, the theory can go just so far; eventually, the real ability to swim must come from actually swimming. It is the same way in problem solving: Students must solve problems! The teacher should try to find problems that are of interest to the students. Listen to them as they talk; they will often tell you about the things in which they are interested. (Problems derived from television and sports always generate enthusiasm among students.)

PROBLEM The weight of a $1 bill is 1 gram. A basketball player's salary is $1 million. The player insists on being paid in $1 bills. Could the salary fit in a suitcase? Could the player carry the suitcase? What if he had decided to accept $20 bills instead?

Discussion Professional athletes and their large salaries are common topics for high school students. This problem fits nicely into their world, so they are motivated to solve it. Here, the first question is open-ended. What size suitcase would be needed? The second question can be answered more easily: $1,000,000 = 1,000,000 grams = 1 metric ton—a pretty heavy load. If the player accepts $20 bills instead, we reduce the weight to approximately 110 pounds, which is manageable.

PROBLEM How far does the needle on a record player travel when a person plays a 12-inch LP record?

Discussion Most teenagers have record collections, but few have even considered this question. At first reading, it would seem that sophisticated mathematics would be required. Closer examination reveals that the needle only travels the sum of the widths of the grooves. This is approximately 3 inches. It would be interesting to draw the path of the stylus as it moves. Notice that the arm that holds the needle is itself the radius of a larger circle; thus, the needle's path is really an arc of this larger circle.

ACTIVITY Take several exercises from the students' textbook. Encourage the students to rewrite the problems, changing the setting of each problem to one that is more interesting to them. Emphasize that the problem must contain the same data as the original.

Discussion One effect of this activity will be to force the students to decide what the problem is really all about. At the same time they will be engaged in creating problems similar to the original, but more interesting to them.

Another way to encourage your students to solve problems is to stop occasionally to analyze what is being done and why the particular processes were undertaken in a particular manner. Focus the students' attention on the larger issue of a general strategy as well as on the specific details of the particular problem at hand. If difficulties arise, make yourself available to *help* students; *do not* solve the problems for them.

ACTIVITY Present students with problems that do not contain specific numbers. Ask them to discuss the problem, provide appropriate numbers, and tell how they would find the answer.

EXAMPLE: Georgia drove from her house to her grandmother's house to spend the weekend. How much did she spend for gas?

EXAMPLE: In a recent competition, the four-member Overbrook relay team ran a race. What was the average time for each runner?

3. Teach students how to read a problem.

As we have said before, every problem has a basic anatomy, consisting of four parts: a setting, facts, a question, and distractors. (In some introductory problems, distractors should be omitted.) Since most problems that students are asked to solve in school are presented to them in written form, proper reading habits are essential. Students must be able to read with understanding.

Fundamental to any problem is an understanding of the setting. If the setting is unfamiliar to the students, it will be impossible for them to solve the problem. Some time should be devoted to merely having the students relate what is taking place in a problem setting. Have them state this in their own words. Asking leading questions may help.

PROBLEM Jane and Susan are the first two people in line to buy their tickets to the rock concert. They started the line at 6:00 A.M. Every 20 minutes, 3 more people than are in the line at that time arrive and join the line. How many people will be in the line when tickets go on sale at 9:00 A.M.?

Discussion This problem is not simple. A careful analysis of the action must be made. What is happening in the line? At 6:00 A.M., there are just 2 people in line. At 6:20 A.M., 5 arrive—3 more than the 2 already in line. At 6:40 A.M., 10 arrive—3 more than the 7 in line at that time. This process continues until 9:00 A.M.

Many activities should be used to help students sharpen their ability to read critically and carefully for meaning. One such technique is to have the students underline or circle words that they consider to be critical facts in a problem. Discuss these words with the class. Have students indicate why they consider these particular words to be critical.

PROBLEM Scott has two dogs. Together they weigh 120 pounds; Koko weighs twice as much as Charcoal. How much does each dog weigh?

Discussion In this problem, the critical facts are "Koko weighs twice as much as Charcoal," and "together they weigh 120 pounds."

ACTIVITY Write a problem on a slip of paper. Have one student read the problem silently, put it away, and then relate the problem in his or her own words to the rest of the class. In this way, students often reveal whether they have found facts that are really important to the solution of the problem, or whether they have missed the point entirely.

ACTIVITY Show a problem on a transparency on the overhead projector. After a short period of time, turn the projector off and have the class restate the problem in their own words.

Since many words have a special mathematical meaning that is different from the everyday meaning, the class should discuss a list of such words, together with their various meanings. A beneficial project is to have the students compile a "dictionary" in which each word is defined both mathematically and in other contexts.

ACTIVITY Discuss the different meanings of the following words:

times
difference
pound
count
volume
foot
face
chord
prime
order
figure

After the students have gained an understanding of the setting of a problem and have identified relevant, important facts, they must be able to identify the question that needs to be answered.

ACTIVITY Give students a set of problems and have them circle or underline the sentence that tells them what they must find. Note that this "question" may be posed in interrogative form or may be stated in declarative form.

EXAMPLE: Michael ran a mile in 3 minutes, 58.3 seconds. By how much time did he win the race if the second place runner completed the race in 4 minutes flat?

EXAMPLE: Andrea has just planted a new garden. She wants to put a rope around it to keep people from walking on it. Find the amount of rope she will need if the garden is in the shape of a circle whose radius is 10 feet.

ACTIVITY Every problem must have a question in order to be a problem. What's the Question is an activity that requires the student to supply a reasonable question based upon a given situation. Asking them to do this forces them to analyze the situation and interrelate the facts—a necessary skill for problem solving.

EXAMPLE: Ralph bought a loaf of bread for 89¢, 6 oranges for $1.10, a jar of tomato sauce for $1.79, and a box of pasta for 63¢.
Make this a problem by supplying an appropriate question.

EXAMPLE: Howard and Donna drove to Georgia from their home, a distance of 832 miles. On the first day they drove 388 miles and bought 38 gallons of gasoline.
Make this a problem by supplying an appropriate question.

The information necessary to solve a problem sometimes appears in verbal form, and other times in picture form. Give the students activities similar to the ones that follow to help them determine which facts in a problem are important.

ACTIVITY Examine the scene shown in Figure 3-1. Then answer the questions.

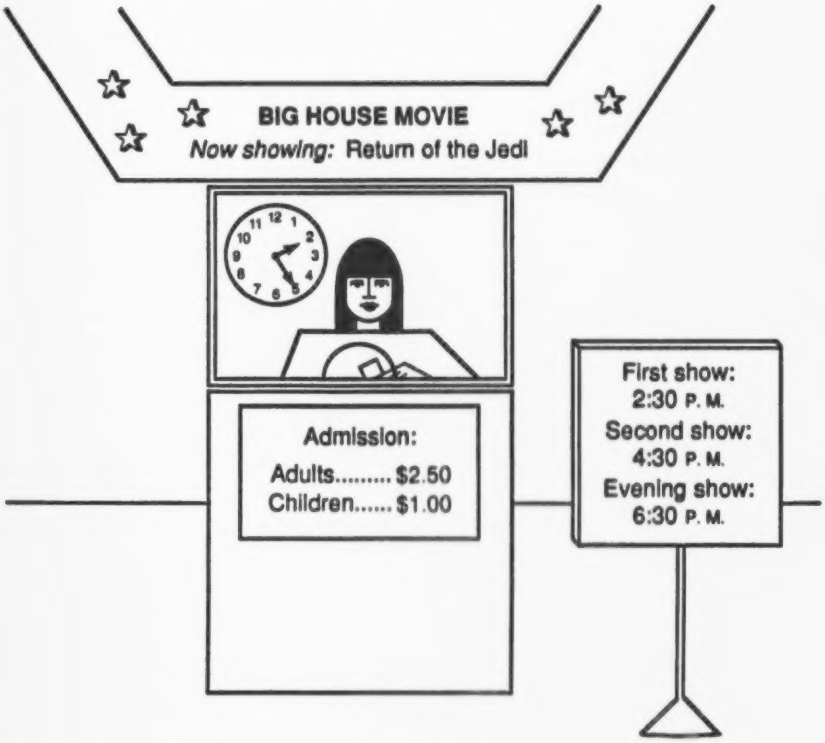

BIG HOUSE MOVIE
Now showing: Return of the Jedi
12
1
2
3
4
5
6
7
8
9
10
11
Admission:
Adults......... $2.50
Children...... $1.00
First show:
2:30 P. M.
Second show:
4:30 P. M.
Evening show:
6:30 P. M.

Figure 3-1

1. An adult wants to see the show. How much does her ticket cost?
2. Has the show started yet?
3. What is the length of the show?
4. Mr. Gomez is taking his daughter to the movies. How much does he spend for the two tickets?
5. What is the name of the picture being shown?
6. Mrs. Spenser took her two children to the movies. How much did she spend?
7. Mrs. Johnson spent exactly $10 for tickets. How many people were in her party?
8. Could there have been exactly 3 adults in Mrs. Johnson's party?
9. What is the maximum length of time someone would have to wait between shows?

ACTIVITY Read the following paragraph. Then answer the questions.

Mr. and Mrs. Rogers and their three children went to the movies to see "Gulliver's Travels." Tickets were $5 for adults and $3 for children. The show lasted 2½ hours. They left the theater at 4:00 P.M.

1. What was the name of the movie they saw?
2. How much do adult tickets cost?
3. How much do children's tickets cost?
4. How long did the show last?
5. How many members of the Rogers family went to the movies?
6. At what time did the show end?
7. At what time did the show begin?

Most of the questions in these two activities could have been answered *directly* from the statements in the text or the facts in the drawing. However, some of the questions required some computation or use of inference. To many of us who are more experienced than the students, these inferences may be immediate. It is most important that the students learn to differentiate between what is given directly and what can be extrapolated from the given information.

ACTIVITY Read the following paragraph. Then answer the questions that follow. Decide whether the information used to answer the question came directly from the paragraph or from some inference and/or computation that you made.

In the Watkins family, there are four children. Cynthia is six years old. Her twin sister, Andrea, is three years older than her brother, Matthew. Their brother James is the oldest, and is three times as old as Matthew.

1. How many children are in the Watkins family?
2. How old is Cynthia?
3. Who is the oldest child?
4. Who is the youngest child?
5. How old is Andrea?
6. How old is Matthew?
7. How old is James?

Reading a problem also means being able to discriminate between necessary and unnecessary information. In many cases, information is put into a problem to serve as a distractor. In other cases, necessary data may have been omitted. We strongly recommend that students be given activities that will enable them to distinguish between necessary and superfluous data, as well as to determine when there is insufficient data to solve the problem. Students should be asked to supply the necessary facts when they are missing from a problem.

ACTIVITY Give the students a problem in written form. Include one piece of extra information. Tell the students to cross out what they think is unnecessary. Have them read what remains. Can they now solve the problem? Repeat the activity, but add a second distractor to the problem. Then have them cross out both pieces of extraneous data.

EXAMPLE: On a trip to the mountains, Janet used 8 gallons of gasoline to cover the 175-mile trip. If gasoline costs $1.19 a gallon, what was the miles per gallon (MPG) for her trip?

Discussion The students should cross out the fact that gasoline costs $1.19 a gallon. The problem deals with miles (distance) per gallon (number of gallons).

EXAMPLE: Long playing records (LPs) cost $8.00 each at the big sale. Compact discs (CDs) cost $10.00 at the sale. Audio tapes cost $4.00 each. If Mike spent $140 for LPs and CDs at the sale, how many of each did he buy?

Discussion The students should cross out the price of the audiotapes ($4.00 each). Notice that several different answers are possible to this problem.

ACTIVITY Give the students a problem in written form, omitting one piece of necessary information. Have one student identify what is missing. Then have a second student supply a *reasonable* fact so that the problem can be solved. Now have the class solve the problem.

EXAMPLE: The Tigers beat the Hawks by 4 runs in the high school championship baseball game. What was the total number of runs scored in the game?

Discussion Students should recognize that there is an insufficient amount of information to answer the question. The number of runs scored by one of the teams must be supplied. Be certain that the number they give is reasonable.

ACTIVITY Divide the class into groups of four students. Have two students in each group create a problem with insufficient information, and the other two create a problem with excess information. Have each group challenge another group to solve their problem. In the first case, they must identify what is missing and supply reasonable data. In the second case, the group must remove the excess information.

ACTIVITY Problem Reader–Problem Solver is an activity that helps to sharpen students' ability to read and comprehend a problem quickly and accurately. It is particularly effective in helping students ascertain the important elements of a problem. Divide the students into groups of four. One pair of students on each team is designated as the Problem Readers, while the other pair is designated as the Problem Solvers. The Problem Solvers close their eyes while a problem is displayed via the overhead projector for about 30 seconds. (The time will depend upon the ability of the students and the difficulty level of the problems used.) During this time, the Problem Readers may take any notes or make any drawings they deem necessary. The problem is then taken off the screen, and the Problem Readers present the problem (as they saw it) to their partners, who must solve the problem. The Problem Readers and the Problem Solvers then reverse roles and play the game again.

ACTIVITY Divide the class into teams of four students. Tell the teams that they are going on a mathematical Scavenger Hunt. Each team must find problems in their textbooks as asked for on the list of items they will receive. Present each team with the same list of questions, similar to the following:

1. Find a problem that has a setting in a supermarket.
2. Find a problem that deals with sports.
3. Find a problem where the answer is an amount of money.
4. Find a problem where the answer is in hours.
5. Find a problem that contains too much information.
6. Find a problem that contains insufficient information.
7. Find a problem that involves geometry.
8. Find a problem where the answer is in square feet.

ACTIVITY Divide your class into groups of four students. Have the first person in each group create a simple, one-stage problem. In turn, each member of the group should (a) add or modify a fact, (b) add or modify a distractor, or (c) add or modify the question. Have each group solve the problem after each modification has been made. Then call "Time" after a reasonable period of time has elapsed. Each group should now present its current version of the problem in turn to the class for solution.

EXAMPLE:
Student #1: I bought 3 T-shirts at $6.95 each. How much did I spend for the T-shirts? (Answer: $20.85.)
Student #2: I bought 3 T-shirts at $5.95 each. How much did I spend for the T-shirts? ($17.85.)
Student #3: Last Monday, I went to the beach, where I bought 3 T-shirts for $5.95 each. I gave the sales clerk a $20 bill. How much change did I get? ($2.15.)

Discussion As each person modifies the problem, the group must determine what effect the modification has on the problem. This sharpens their ability to analyze the relationship between facts, distractors, and questions.

4. Require your students to create their own problems.

Nothing helps students understand problems better than having them make up their own. In order to create a problem, students must know the anatomy of a problem. They must relate setting, facts, questions, and distractors. We know that students create excellent problems. In fact, their problems are usually more relevant (and often more complex) than the ones found in textbooks.

However, in order to generate problems, students need something to write about. Presenting a suitable stimulus will aid students in this endeavor.

ACTIVITY Provide students with a set of answers such as 15 books, $7.32, 8 miles, 12½ gallons, and 150 square feet. Have them create problems for which each of these is the correct answer.

ACTIVITY In an algebra class, provide the students with an equation or a pair of equations. Ask them to write a problem for which the solution to the equation(s) is the answer to the problem.

Discussion We might give them the equation

$$2x + 3 = 17$$

A suitable problem for this equation might be:
Mary and John went to the circus. They bought 2 tickets, and spent $3 on popcorn. If they spent $17 in all, how much did each ticket cost?
A pair of equations such as

$$\begin{aligned} x + y &= 12 \\ 2x + 3y &= 32 \end{aligned}$$

might yield the following problem:
Randy has two-wheel bicycles and three-wheel tricycles in his store. He needs 32 tires to change all the tires on the 12 vehicles. How many bicycles and how many tricycles does he have?

ACTIVITY Show pictures that have been taken from old magazines, old catalogues, newspapers, old textbooks, etc. Have the students make up problems based upon the pictures.

Discussion This activity helps students decide on numerical data that makes sense, for they must use realistic numbers in their problem designing. At the same time, this activity helps students relate mathematical problems to their other subjects such as social studies, language arts, and science, as well as to real life.

ACTIVITY Ask your students to write a "menu problem." That is, given the following menu, write a problem about it.

Hot dog		.95
Hamburger		1.25
Pizza (slice)		.85
Tuna sandwich		1.25
Grilled cheese sandwich		1.05
Apples		.25
Bananas	2 for	.45
Milk (white)		.30
Milk (chocolate)		.35
Candy bars	2 for	.65

Discussion At first, many students will probably write a problem that merely lists an order for two or more items and asks for the cost.

EXAMPLE: Stan bought a hamburger and a chocolate milk. How much did he spend for his lunch?

> Some students may give a higher level of problem, in which the amount of change from a large bill is asked for.
>
> EXAMPLE: Arlene had a grilled cheese sandwich, an apple, and a white milk. How much change did she receive from $5.00?
>
> A third level problem might involve stating the total amount spent and asking for what could have been purchased.
>
> EXAMPLE: George spent $2.60 for his lunch. What might he have bought?
>
> Students may be surprised when multiple answers appear. Adding the tax may complicate the problem even further.

Select several of the students to present their problems to the class. Have the entire class solve the problems. Sharing problems that have been written by other students should be an integral part of classroom procedure. The fact that the problems have been designed by classmates usually heightens interest in solving them. These problems may simply be variations of other problems that students have seen, or they may be entirely original creations.

As the students gain experience in creating their own problems, the problems will become more and more sophisticated. There will be some with insufficient information and some with excess information. This is highly desirable, because the problem-solving process is what should be stressed. Problems that appear in textbooks often emphasize one particular skill or operation. On the other hand, solution of these student-generated problems frequently involves extraneous data and possibly more than one level of operation.

5. Have your students work together in small groups.

Team problem solving and group brainstorming are viable techniques in the business, scientific, and professional communities. Rarely does any one person solve major problems alone. While the final decision does fall on one person, the group input is an integral part of the problem-solving process. A student's ability to help in this group process is highly desirable.

Brainstorming is not ordinarily used in the traditional classroom setting. Students have had no training or experiences in the sharing of ideas or the interaction required. Thus, the classroom teacher must provide guidance and practice in the particular skills involved in sharing ideas. The teacher should keep in mind that in brainstorming:

1. There should be no evaluation of any kind. People have a tendency to become defensive of their own ideas. In addition, other people may not participate if they feel that they are to be formally judged.
2. All the students are encouraged to allow their imaginations to run rampant.
3. Participants are encouraged to put forth as many ideas as possible.
4. Everyone is encouraged to build upon the ideas or to modify the ideas of others.

The interaction provided by cooperative students will help them learn to modify one another's thinking and to clarify their own. They will also learn to express their thoughts more clearly by the use of precise language, especially mathematical terminology. Students will find it difficult to communicate with others unless they use language that every member of the group can agree upon.

The group may want to focus on a problem-solving situation as a series of motion picture frames. The sequential nature of this image helps students decide what comes first, what comes second, and so on. With contributions from various members of the group, the class can develop a sequence of activities to solve the problem.

Consideration must be given to the forming of the small groups. This must be done quite carefully if the group is to function effectively. We suggest a heterogeneous grouping, so that the weaker students can be assisted in the task by the stronger students. This peer interaction has been shown to be very effective in problem solving.

ACTIVITY "Linking-Thinking" is an activity that will introduce students to brainstorming. Divide your class into groups of 5 or 6 students. Each group should be provided with a trio of what appear to be totally unrelated words, such as:

(a) pencil—apple—football
(b) elephant—clock—ice cream cone
(c) sunglasses—flashlight—kumquat
(d) radio—cereal—computer

Notice that the words in each set do not appear to have anything in common. It is the group's task to create a connection (the link) between the elements of each set. After a few minutes, have each group present and discuss the connection they have formulated.

ACTIVITY Using the Linking-Thinking activity described above, assign each group the task of writing a problem in which each of the 3 objects named plays an integral role. Have each group, in turn, present their problem to the entire class for solution.

ACTIVITY "Notice" is a quiz that is administered to students in groups. The entire group must arrive at one single True or False decision for each statement. Here are some sample statements:

(a) The Statue of Liberty uses her right hand to hold the torch. (True)
(b) A record on a turntable will turn clockwise. (True)
(c) Page 82 of a book is a right-hand page. (False)
(d) Most pencils have eight sides. (False)
(e) Q is the only letter that is missing on a telephone. (False).

ACTIVITY "What Action Would You Take?" is an activity designed to force a group to make a decision. The decision must be the result of deliberation by the entire group. Present the group with a problem similar to the ones that follow. Have each group decide what action should be taken.

EXAMPLE: A truck filled with 10,000 straw hats has gone off the road and is stuck in a muddy ditch. Explain how you would use your cargo to get the truck back on the road.

EXAMPLE: Your group is assigned the task of determining the most expensive item in the supermarket. What action would you take?

EXAMPLE: Your group is leading an exploration of one area of the moon's surface. You come to a giant crater and build a suspension bridge across it. The bridge can hold only 1,600 pounds. You have a lunar rover that weighs 1,800 pounds. Each wheel weighs 60 pounds, and the average weight of each member of your group in his or her space suit is 180 pounds. How will your group get the lunar rover across the bridge?

EXAMPLE: Your group has been given $1,000,000 to spend in a 24-hour period. How would you spend the money? (You may buy only one of each item selected).

We do not advocate that students solve all problems in groups. However, the group process provides an excellent opportunity for students to express their own opinions and to appreciate the contributions made by others. Sometimes, the very student who is reluctant to make a contribution in a full classroom setting is more than willing to participate in a small group. This setting also permits you, the teacher, to gain a different perspective of individual class members.

6. Encourage the use of drawings.

A drawing is a paper-and-pencil simulation of the action described in a problem. Having your students consistently make drawings will enable them to convert a verbal situation into a visual representation. This pictorial representation forces the student to remove the distractors, recognize the facts, and understand the relationships that are inherent in the problem. It also represents the intermediate stage between the concrete and the abstract.

Emphasis should be placed upon neatness, accuracy, and proper scaling. Perpendiculars should look as if they form 90-degree angles; isosceles triangles should have 2 congruent sides. Careless drawings can lead to improper thought processes and faulty conclusions.

ACTIVITY Distribute a series of problem situations that can be described by a drawing. Have each student make a drawing to illustrate the action. Discuss the various drawings they have made.

EXAMPLE: Three logs, each 3 feet in diameter, are tightly banded together with a steel strap. Find the length of the strap.

Discussion A correct interpretation of the problem is revealed by the drawing shown in Figure 3-2.

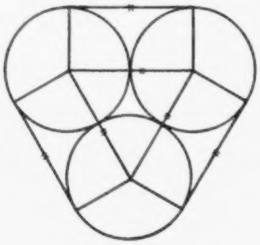


Figure 3-2

Without a drawing, this problem is extremely difficult to solve. A drawing readily reveals that the strap consists of two parts; a straight-line section that equals the perimeter of the triangle connecting the centers and a curved part consisting of three 120° arcs.

EXAMPLE: The Blue Pelican sets sail from its home port at a bearing of 120°. At the same time, the White Swan starts from its home port, 50 miles due south of the home port of the Blue Pelican, at a bearing of 30°. The two ships meet after 2 hours. How far is the Blue Pelican from its starting point when they meet?

Discussion Again, a drawing leads the student to the solution. The paths of the two ships along with the location of the two ports reveals a 30°–60°–90° right triangle (Figure 3-3).

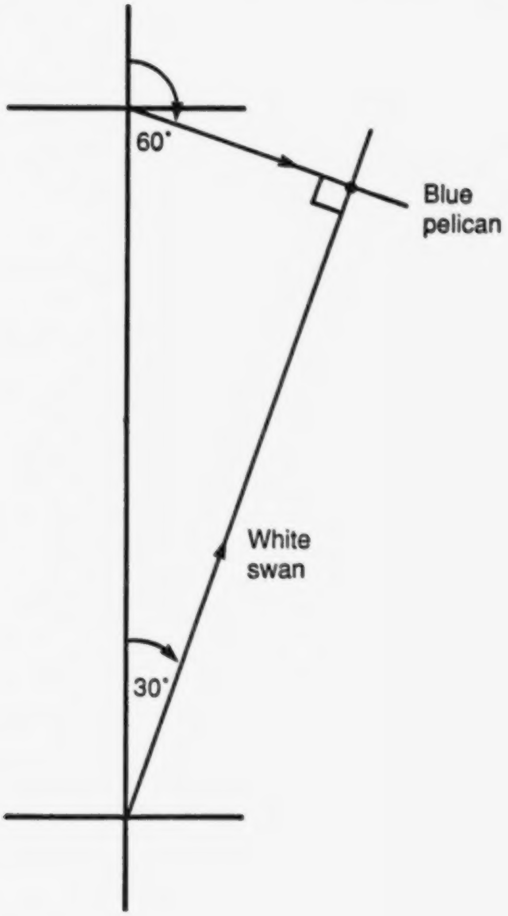


Figure 3-3

ACTIVITY Distribute several drawings that illustrate problem situations. Have your students make up a problem for which the given drawing is appropriate. Discuss the problems with the entire class.

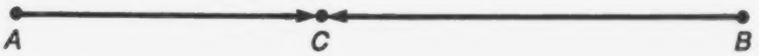


Figure 3-4

Discussion One problem that Figure 3-4 suggests would be the following. Two cars are driving towards each other at speeds of 30 miles per hour and 60 miles per hour. If they leave their respective locations at 10:00 and they are 135 miles apart, at what time do they meet?

7. Have your students flowchart their own problem-solving process.

Problem solving is a thought process. Students must be able to analyze their own thinking as they develop their individual ability to problem solve. They must be able to identify the components of their own problem-solving process. To assist in this, we suggest that they construct a flow chart showing the steps they utilize. At first, the flow chart might be a simple one, like the one shown in Figure 3-5.

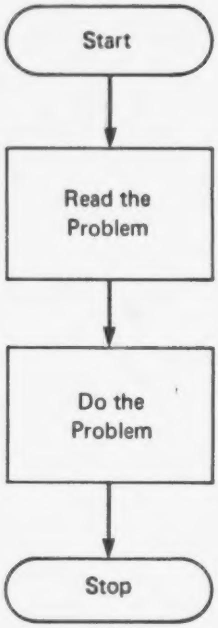


Figure 3-5

However, as the students experience more and more problem solving, their flowcharts will become more complex, like the one in Figure 3-6.

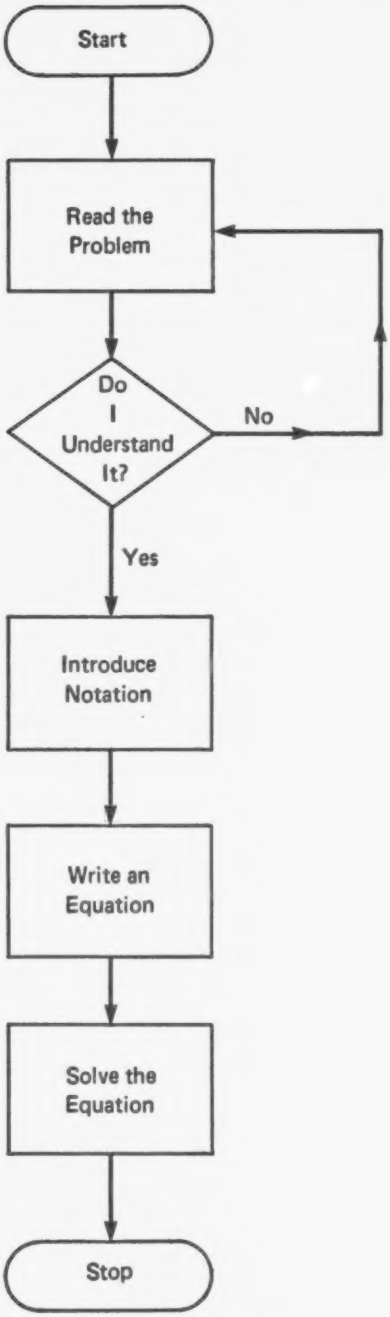

Start
Read the Problem
Do I Understand It?
No
Yes
Introduce Notation
Write an Equation
Solve the Equation
Stop

Figure 3-6

Notice that the students are now beginning to ask themselves questions. This is extremely important in the problem-solving process.

Finally, as the student becomes more skilled and develops a more sophisticated procedure, the flow chart should resemble the one shown in Figure 3-7 on p. 68.

The preparation of a flowchart for problem solving is an extremely valuable procedure for both students and the teacher. It will help the students organize their own thoughts. (We feel that anyone who cannot flowchart a process does not really understand that process.) At the same time, it will provide the teacher with a chance to examine the problem-solving process as the students perceive it. It is a visual example of what the students are thinking as they solve a problem.

8. Suggest alternatives when the present approach has apparently yielded all possible information.

It often happens that a chosen strategy fails to provide an answer. Good problem solvers then seek another approach, but other students repeat the same method of attack. This usually leads to a dead end. This repetition is not unusual, nor is it unexpected. In fact, the condition is referred to as the *einstellung* effect. This is a predisposing condition or mindset that usually leads to the same end over and over again, by blocking out any kind of variable behavior. This mindset must be changed and some other approach undertaken if the student has not successfully resolved the problem. Many students give up or follow the same path again and again. It is at this point that some teachers err; they often direct the students through the most efficient path to the solution, rather than allowing further exploration. The teacher should guide the additional exploration by pointing out facts and inferences that might have been overlooked.

ACTIVITY Present a problem to the class. Have the students work in groups and attempt to solve the problem in as many different ways as possible. Have each group present its solutions to the entire class. See which group can find the greatest number of solutions.

PROBLEM In Figure 3-8 on page 69, AB and CD are perpendicular diameters of length 8 centimeters, intersecting at O. Any point, F, is selected on arc AC. Perpendiculars FG and FH are drawn to OA and OC respectively. Find the length of segment GH.

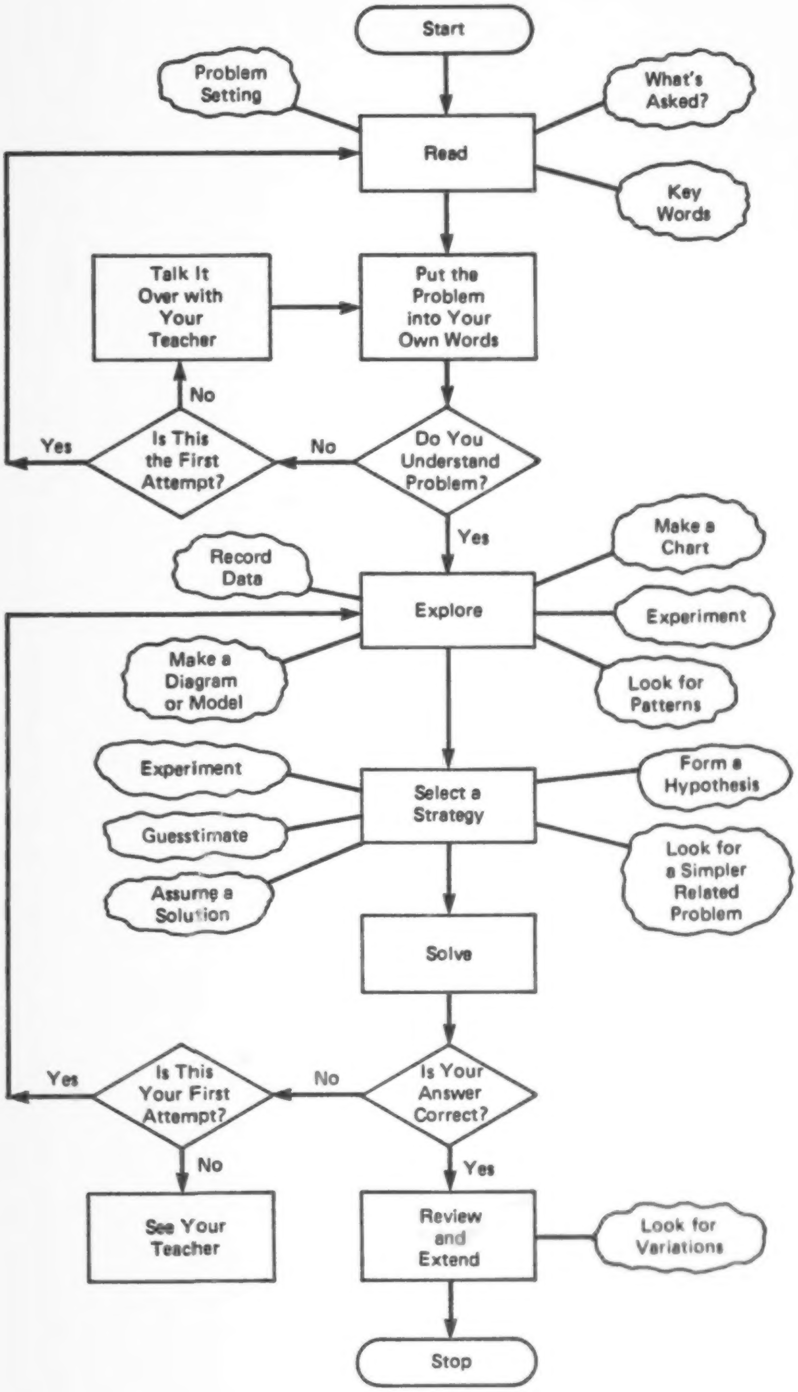

Start
Problem Setting
What's Asked?
Read
Key Words
Talk It Over with Your Teacher
Put the Problem into Your Own Words
No
Yes
Is This the First Attempt?
No
Do You Understand Problem?
Yes
Record Data
Make a Chart
Explore
Experiment
Make a Diagram or Model
Look for Patterns
Experiment
Select a Strategy
Form a Hypothesis
Guesstimate
Assume a Solution
Look for a Simpler Related Problem
Solve
Yes
Is This Your First Attempt?
No
Is Your Answer Correct?
No
Yes
See Your Teacher
Review and Extend
Look for Variations
Stop

Figure 3-7

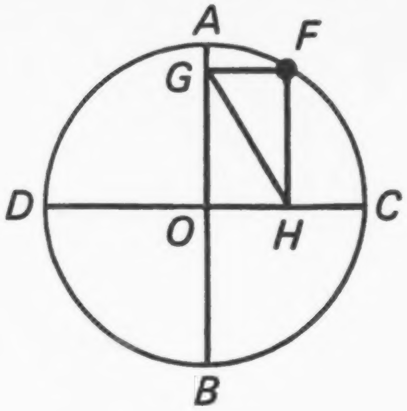


Figure 3-8

Discussion In solving this problem, most students have a mindset that suggests the use of the Pythagorean Theorem in triangle *GHO*. Once they have exhausted all the information in the problem, however, the teacher might suggest to them that they consider the rectangle *FGOH* and *both diagonals*.

PROBLEM David is serving a fresh apple pie to his three brothers and himself. His oldest brother, Mike, bets his share that David cannot cut the pie into four equal portions without lifting his knife or going back over a cut. David thinks for a minute and then wins the bet. How does he do this?

Discussion David cuts the pie with a "figure eight" cut, passing through the center of the pie, as shown in Figure 3-9.

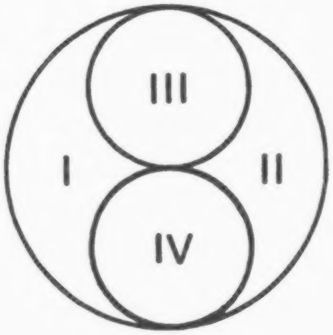


Figure 3-9

Since the radius of the original pie is r, its area equals πr^2. Thus the area of each small circle (labeled III and IV) is

$$\pi \cdot \left(\frac{r}{2}\right)^2 \quad \text{or} \quad \pi \frac{r^2}{4}$$

which is one-fourth of the original pie. By symmetry, portions I and II are equal. Thus each piece (I, II, III, and IV) is one-fourth of the original pie. Once students rid themselves of the mindset that the cuts must be straight lines, the problem can be solved most readily.

PROBLEM The sum of two numbers is 7 and their product is 25. Find the sum of their reciprocals.

Discussion The immediate reaction to this problem would be to let x and y represent the two numbers and form the two equations

$$x + y = 7$$
$$xy = 25$$

which are then solved simultaneously. As the equations are solved, we are led through the quadratic formula, which yields complex roots. In addition, students must rationalize the denominators of two expressions in the form $1/(a + bi)$ and $1/(a - bi)$.

If students have been taught to search for alternate solutions, they might work backwards. Notice that the original problem asks for the sum of the reciprocals, not the value of the two numbers. Thus we wish to find the value of $\frac{1}{x} + \frac{1}{y}$, which is $\frac{x + y}{xy}$ or $\frac{7}{25}$.

PROBLEM A pirate ship at point A in Figure 3-10 is 50 meters directly north of point C on the shore. Point D, also on the shore and due east of point C, is 130 meters from point C. Point B, a lighthouse, is due north of point D and 80 meters from point D. The pirate ship must touch the shoreline and then sail to the lighthouse. Find the location of point X on the shoreline so that the path from A to X to B will be a minimum.

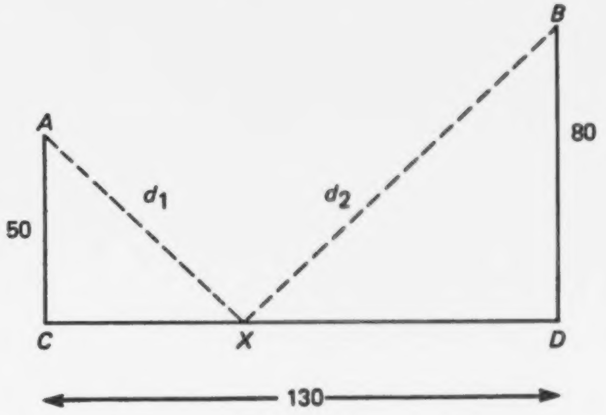


Figure 3-10

Discussion If students decide to use a calculator, they can use a series of successive approximations, as follows:

CX	XD	d_1	d_2	$d_1 + d_2$
0	130	50.0	152.6	202.6
20	110	53.9	136.0	189.9
40	90	64.0	120.4	184.44
50	80	70.7	113.1	183.8←(approximate
60	70	78.1	106.3	184.41 location of X)
80	50	94.3	94.3	188.6
100	30	104.4	85.4	189.8
130	0	139.3	80.0	219.3

Although these calculations do yield an approximate location for point X, the teacher might offer the following suggestions to students to help them "exactly" locate point X. The term "minimum distance" is the same as "shortest path." Students should realize that the shortest path between two points is a straight line. Thus, if we reflect segment *BD* its own length to *B′* so that *B* and *B′* are on opposite sides of *CD*, we can draw *AB′*, as in Figure 3-11 on page 72. Since *AB′* is a minimum, and since triangle *BDX* is congruent to triangle *B′DX*, segment *BX* (or d_2) is equal to segment *B′X*. Thus point *X* is the exact location we wish.

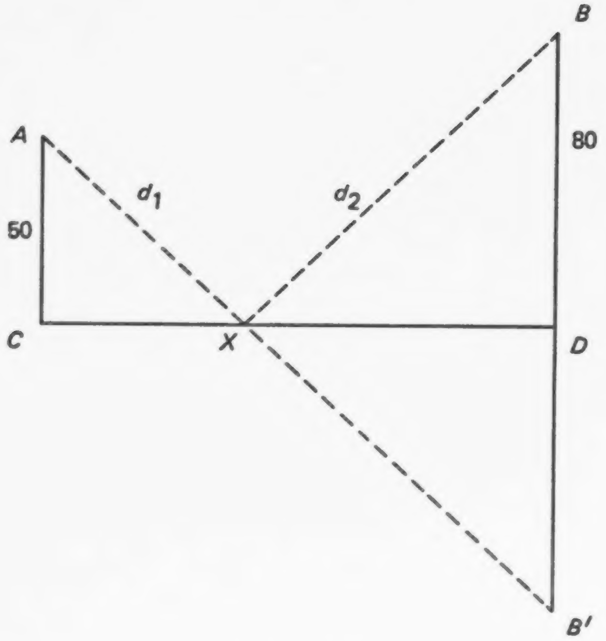


Figure 3-11

PROBLEM In an office, there are two square windows. Each window is 4 feet high, yet one window has an area that is twice that of the other window. Explain how this can take place.

Discussion The usual way to envision a square window is with the sides parallel to and perpendicular to the floor. However, if we consider a window that has been rotated through 45°, we can readily see the explanation. (See Figure 3-12.)

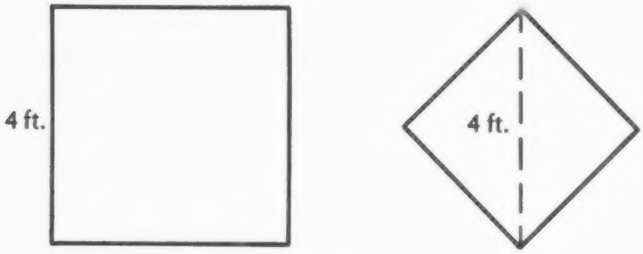


Figure 3-12

PROBLEM Find the sum of all proper fractions whose numerators and denominators are positive integers and whose denominators are less than 100.

Discussion At first, this problem seems overwhelming, and students might be tempted to give up without trying. However, careful analysis and organization will reveal some interesting relationships that will lead to the answer.

$$\begin{array}{lllll}
\frac{1}{2} & & & & = \frac{1}{2} \\[1em]
\frac{1}{3} & + \frac{2}{3} & & = \frac{3}{3} = 1 & = \frac{2}{2} \\[1em]
\frac{1}{4} & + \frac{2}{4} & + \frac{3}{4} & = \frac{6}{4} = 1\frac{1}{2} & = \frac{3}{2} \\[1em]
\frac{1}{5} & + \frac{2}{5} & + \frac{3}{5} & + \frac{4}{5} = \frac{10}{5} = 2 & = \frac{4}{2} \\
 & \vdots & & & \\
\frac{1}{99} & + \frac{2}{99} & + \frac{3}{99} + \cdots & + \frac{98}{99} = 49 & = \frac{98}{2}
\end{array}$$

Thus the sum may be found by adding

$$\left(\frac{1}{2} + \frac{2}{2} + \frac{3}{2} + \frac{4}{2} + \frac{5}{2} + \cdots + \frac{98}{2}\right)$$

When students are stuck, you might suggest that they look back at other problems they have solved in the past that were similar to the problem under consideration. This might lead to some ideas of what to do. Even suggesting what might be done at a particular point is sometimes in order. Thus you could suggest to students that "it might be a good idea" if they

1. made a guess and checked it
2. tried a simpler version of the problem
3. made a table
4. drew a diagram
5. used a physical model
6. used a calculator
7. worked backwards
8. looked for a pattern
9. divided the problem into several parts and solved each
10. used logical thinking
11. examined a similar problem

Even encouraging students simply to pause and carefully reflect on the problem is a good technique to try when they are totally stymied.

9. Raise creative, constructive questions.

Due to pressures of time, the teacher often misses the opportunity of expanding the discussion beyond the scope of the immediate problem; it seems more efficient to take the students directly to the answer with a polished solution. Instead, you should ask questions that will provide the students with guidance and direction yet allow for a wide range of responses. Give them time to think before they respond to your questions. Research indicates that the average teacher allows only up to 3 seconds when students respond to a question. Problem solving is a complex process; you must allow time for reflection. Don't rush your questions.

In trying to guide your students through a solution, use open-ended questions frequently. "How many . . . ?", "Count the number of . . .", and "Find all . . ." are all non-threatening questions that lead to successful student responses.

PROBLEM How many triangles are there in Figure 3-13?

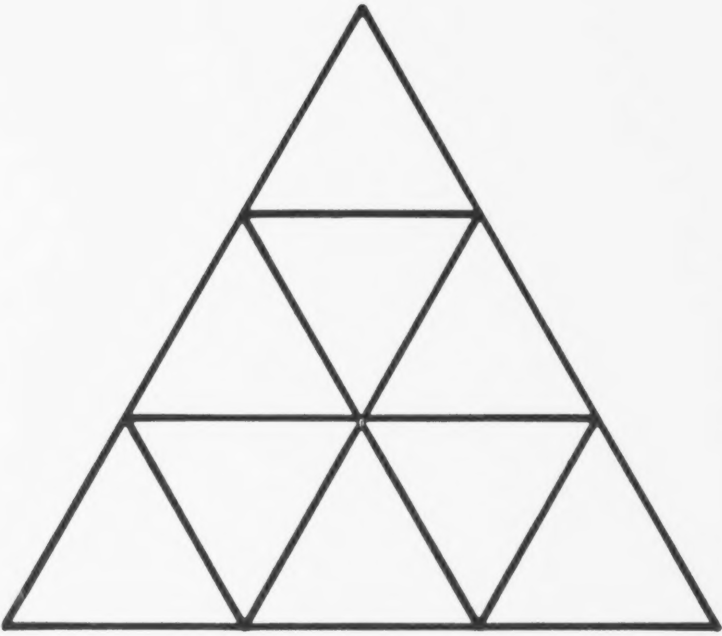


Figure 3-13

Discussion Some students may answer, "Nine!" (the number of small triangles in the figure). While this may be a good first response, there are several questions you should ask to expand the student's perception. For example, "What about the large triangle itself?" (Now we have 10 triangles.) "Are there any other hidden triangles?" "What happens when you cover up the bottom row of triangles?" "Could you label all the points?" "Name some of the triangles you see."

Throughout the problem-solving process, let your questions cause the students to reflect back upon the problem solution. Too often we tend to turn away from a problem that has been "solved" (i.e., for which an answer has been found), in order to move on to the next problem. Thus we miss a chance to glean extra values from our energy. Examine the entire solution carefully; ask questions about key points. Ask many "What if . . ." questions. Ask students "What new question does this suggest?" or "How might I ask this question in another way?"

Other questions that you might ask include:

1. Do you recognize any patterns?
2. What is another way to approach the problem?
3. What kind of problem does this remind you of?
4. What would happen if . . .
   the conditions of the problem were changed to . . . ?
   the converse were asked?
   we imposed additional conditions?
5. What further exploration of this problem can you suggest?

Be careful that

1. Your question is not changed or altered while the students are considering it;
2. You give the students ample time before repeating the question.
3. You do not answer your own question until you are certain that the students have finished their responses. Perhaps an additional hint or comment might lead them in the right direction.

10. Emphasize creativity of thought and imagination.

In a risk-free classroom atmosphere, students should be able to express their thoughts openly. You should not reject "way out" answers if they show some thought on the part of the student. Problem solving requires creating, developing, and linking ideas and thoughts.

ACTIVITY A student's response to the question, "How can I divide 25 audio tapes among 3 people?" was, "I'll take 23 of them and give 1 to each of the other 2." Discuss the answer.

ACTIVITY Karen drove from New York to Atlanta in 16 hours, a total distance of 800 miles. How fast did she drive? Discuss this problem.

The two activities suggested above will often yield answers that are quite different from what was expected. Students may interpret the problem in a variety of ways. Some interpretations, although seemingly far-fetched, may be quite plausible when examined carefully. In all cases, you must lead the class in a thorough discussion of *why* these various interpretations are justifiable.

A classic story that illustrates the point of this section is the well-known barometer problem, "How would you use a barometer to measure the height of a high-rise apartment building?" Several plausible, imaginative solutions follow:

1. Measure the barometric pressure at the top and the bottom of the building. Use the appropriate formulae to compute the height of the apartment building.
2. Attach the barometer to a long rope. Lower the barometer to the street and measure the length of the rope.
3. Drop the barometer from the roof of the building and time its fall. Use the appropriate formulae to determine the height of the apartment building.
4. Use the shadows of the building and the barometer, together with the actual size of the barometer. By use of similar triangles and proportion, the height of the building can be determined.
5. Give the barometer to the superintendent of the building in exchange for the needed information.

Puzzle problems can also be used to develop imaginative and creative thinking.

PROBLEM Place the numbers 1, 2, 3, 4, 5, and 6 in the spaces provided in Figure 3-14, so that each side of the triangle adds up to 10.

PROBLEM You have 10 piles of silver coins. Each coin weighs 2 ounces. However, someone replaced all the coins in one of the piles with counterfeit coins, each of which weighs

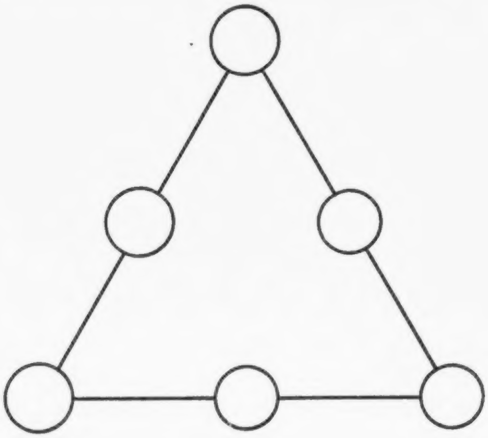


Figure 3-14

1½ ounces. You have a scale, and wish to find the counterfeit pile with only one weighing. How would you do it?

Discussion We can select 1 coin from pile number 1, 2 coins from pile number 2, 3 coins from pile number 3, and so on up to 10 coins from pile number 10. Now, when we weigh these 55 coins, they *should* weigh 110 ounces (55 × 2) if all were legitimate. But this will not happen. By determining the number of half-ounces we are short in our weighing, we have the number of coins and the number of the pile that contains the counterfeit coins.

PROBLEM A banker has 9 gold coins, one of which is counterfeit and weighs less than the other 8. Using a balance scale, how can the banker determine which coin is the light one in just two weighings?

Discussion Let's number the coins 1 through 9. Now we weigh coins 1, 2, and 3 against coins 4, 5, and 6. If the scale balances, the light coin is not among these first six coins. In the second weighing, weigh coin 7 against coin 8. If they balance, the counterfeit coin is number 9. If they do not balance, the counterfeit coin is the lighter one. (If coins 1, 2, 3, and 4, 5, 6 do not balance in the first weighing, repeat the second weighing with two coins selected from the lighter group).

Obtaining the correct answer is not really the reason for proposing these problems. Their solution requires a great deal of ingenuity. The situation encourages all kinds of responses, thus affording the opportunity for students to think creatively.

11. Apply the power of algebra.

If your students have had the benefit of a course in algebra, they are in an advantageous position to become good problem solvers. Many problems can be neatly expressed in algebraic notation, and many of these can be solved by a traditional algebraic treatment. Have the students read the problem and, if possible, express the relationships in algebraic form. Can they form an equation? Can they form a system of equations? Can these be solved? Have the students carry out the algebra and obtain an answer if possible. This gives practice in the fourth step of the heuristic, "solve."

In some cases, the algebraic solution may be quite cumbersome, and another approach or solution may be more artistic or creative. A good problem solver will attempt to do the problem in other ways.

PROBLEM A woman was $\frac{3}{8}$ of the way across a bridge when she heard the Wabash Cannonball Express behind her, approaching the bridge at 60 miles per hour. She quickly calculated that she could *just* save herself by running to either end of the bridge at top speed. What is her top speed?

Discussion Draw a diagram as in Figure 3-15. We see that there are two conditions to consider.

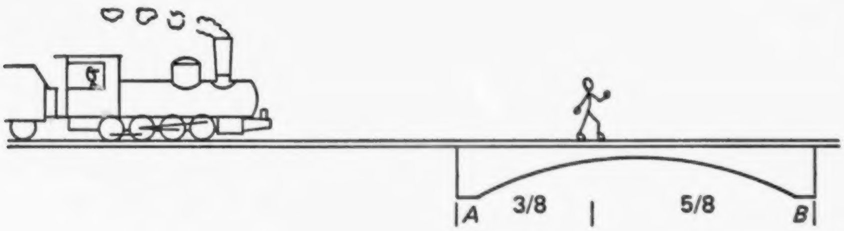


Figure 3-15

Let x represent the distance the train is from point A.

Let y represent the rate at which the woman can run.

Now, since the length of the bridge was not given, we select a convenient length, say 8 units. Then,

$$\frac{3}{y} = \frac{x}{60} \quad \text{(if she runs to point } A\text{)}$$

$$\frac{5}{y} = \frac{x + 8}{60} \quad \text{(if she runs to point } B\text{)}$$

$$\begin{aligned} xy &= 180 \\ xy + 8y &= 300 \\ 8y &= 120 \\ y &= 15 \end{aligned}$$

Her top speed is 15 miles per hour.

Another solution would involve the following reasoning:

> If she can *just* get to either end of the bridge, before the train arrives there, let her run away from the train toward point *B*. When the train arrives at point *A*, she will have covered an additional $\frac{3}{8}$ of the bridge's length, or $\frac{3}{4}$ of the bridge. She can now run the remaining $\frac{1}{4}$ of the bridge in the same time it will take the train to cover the entire $\frac{4}{4}$ of the bridge. Thus, her rate is $\frac{1}{4}$ that of the train, or 15 miles per hour.

Algebra, in addition to being a powerful tool, can be considered a way of thinking. A student who has gained algebraic sophistication can read a problem and express it as a set of algebraic sentences or statements, thus providing a path to an answer. In terms of our heuristics, that student has gone through the process as far as the "solve" step. Whether the student continues with the algebraic solution or uses an alternate process is really of little significance. The student has demonstrated an understanding of the essential parts of the problem. The algebra may be his or her way of summarizing the critical points of the problem. Some students may realize that an answer may be obtained more easily by non-algebraic methods.

12. Emphasize estimation.

There are basically two distinct types of estimation: physical and numerical. *Physical estimation* involves estimating quantities such as the height of a building, the length of a room, the weight of a box, a period of time. *Numerical estimation* is usually used in approximating the result of a computation. This latter type of estimation has become increasingly important with the prominence of the calculator. Students must know if the answers they obtain from their calculators are "in the ballpark."

Estimation is a very elusive skill. There are no specific guidelines that satisfy all demands, nor do most people know what it is they do when they estimate. Each of us develops his or her own particular techniques of estimation over a period of time. When students estimate, a range of answers should be regarded as normal, as long as they are reasonable.

The activities that follow are designed to help students attain some facility with both kinds of estimation skills.

Physical estimation

ACTIVITY Find a book with a large number of pages. Provide the students with a referent by telling them the number of pages in the book. Place a bookmark anywhere in the book. Have the students look at the bookmark and estimate the number of the page it is marking. Try this several times. Have students keep a record of how many times their guesses were within 10 pages of the bookmark.

ACTIVITY Have each student take a textbook with as many pages as possible. Have them place a bookmark where they estimate the middle of the book to be. Have them check their results. Repeat the activity several times, estimating where one-fourth of the book is, where three-fourths of the book is, where page 100 is, and so on.

ACTIVITY Have someone measure the width of the classroom and give this information to the class. This serves as the reference length. Now have the students estimate the length and the height of the room. Compare these estimates. Then actually measure the lengths in question and compare these to the students' estimates. Now ask the students to estimate the diagonal length of the classroom. Check and compare.

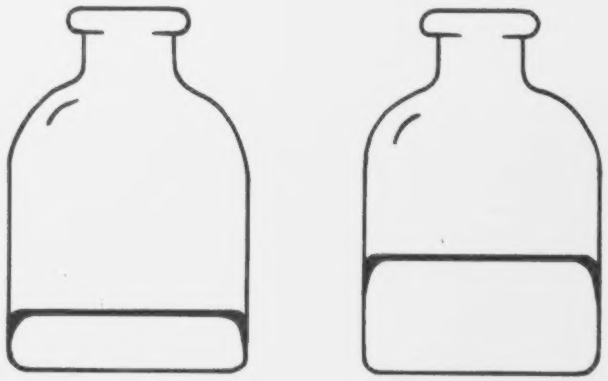
ACTIVITY Once the students have had some experience with linear estimation, bring a long rope into class. Have two students stand at opposite corners of the room, holding the rope. Allow the rope to sag. Ask the students to estimate the full length of the rope.

ACTIVITY Have two students hold a rope tautly across the room. Place a clothespin somewhere along the rope. Ask the students to estimate the lengths of the two segments thus formed. Move the clothespin to various other positions. Each time, have the students estimate the lengths of the two segments. Then have the students attempt to place the clothespin one-fourth of the way, one-third of the way, or at specific distances along the rope.

ACTIVITY Fill a glass pitcher with water. Pour off one glass in front of the class. Permit them to see the water level in the pitcher drop. Ask the students to estimate how many glasses of water remain in the pitcher. Compare their estimates with the actual number.

ACTIVITY Ask the students to close their eyes and guess how long a minute is. Check them with a clock. See if they can come within 15 seconds, then within 10 seconds, then within 5 seconds. Have them record how close they come each time.

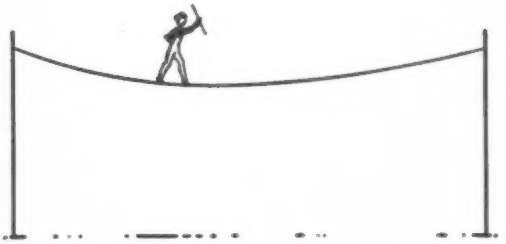
Drawings can also be used to provide students with practice in estimation skills. Figures 3-16 and 3-17 give examples of how to use drawings to provide practice for students. Notice that, in each case, they must establish a reference base.



This jar contains 10 ounces of oil.

How many ounces of oil does this jar contain?

Figure 3-16



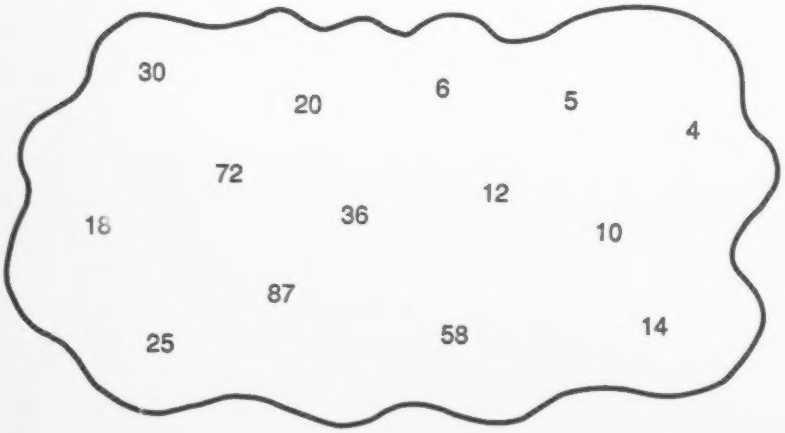
The tightrope walker has walked 20 feet.
How many more feet must he walk to
reach the other side?

Figure 3-17

Numerical estimation

Most problems in mathematics eventually result in some form of computation. This will usually take place as students begin the "solve" step of the heuristic process. It is essential that they be able to determine if their answer is "close." To do this, students should become skilled in numerical estimation.

ACTIVITY Provide the students with an array of numbers similar to that shown below:



Ask the students to select pairs of numbers whose product is approximately 360.

ACTIVITY Have the students select sets of three numbers from the same array, whose sum is 100.

The same type of activity can be used to practice estimation with differences and quotients as well.

ACTIVITY "Thumbs up—thumbs down" is a game setting that provides students with practice in estimating sums of 100. Prepare a series of cards with a two-digit number on each. Select several with sum approximately 100. For example,

$$18 + 95 \qquad 16 + 26 + 37$$
$$58 + 40 \qquad 32 + 41 + 36$$

If the sum is greater than 100, the students should put their "thumbs up." If it is less than 100, have them put their "thumbs down."

The same activity can be repeated with three-digit numbers, using 1,000 as the target answer. It can also be used to estimate products, quotients, differences, and fractional sums.

ACTIVITY Prepare a sheet of "Is It Reasonable?" problems for the students. They are not to solve the problems. Rather, they must decide if the answer given is a reasonable one. Here are some samples you might use.

1. A hurricane is moving northward at the rate of 19 miles per hour. The storm is 96 miles south of Galveston. Approximately how long will it take the hurricane to reach Galveston?

Answer: It will take the storm 5 minutes to reach Galveston. (Not reasonable—it will take approximately 5 *hours*, not 5 minutes.)

2. A cog railroad makes 24 round trips each day up the side of a mountain. On Monday, a total of 1,427 people rode the cog railroad. About how many people rode on each trip?

Answer: About 60 people rode on each trip. (Reasonable.)

3. During the baseball season, 35,112 fans went to a Red Sox–Yankees game one Sunday, while 27,982 fans went

to a Phillies–Braves game that same day. About how many more fans went to the Red Sox–Yankees game?
Answer: About 700 more fans were at the Red Sox–Yankees game. (Not reasonable—it should be about 7,000.)

4. A bread baking company has 48 delivery trucks. Each truck can hold 1,390 loaves of bread when filled to capacity. About how many loaves of bread can all the trucks carry when loaded to capacity?
Answer: About 70,000 loaves. (Reasonable.)

5. Mrs. Arnold bought 21 copies of back issues of *Superman Comics* for $32.75. About how much did she pay for each copy?
Answer: About 75¢ per copy. (Not reasonable—it should be about $1.50 a copy.)

Activities such as these will help your students to improve their ability to estimate answers. Most of all, such practice will encourage them to take a "guesstimate." In fact, this is an important part of the problem-solving process.

13. Utilize calculators.

More than 200 million hand-held calculators have been sold in the United States. The majority of your students probably either own a calculator or have access to one within their immediate household. A teacher can provide conceptual experience long before the student makes the generalization, and concepts can often be extended to the real world through the use of the calculator. The students should perform all of their calculations with a calculator. They can work with problems that are interesting and significant, even though the actual computations may be beyond their paper-and-pencil capacity. The focus can now be on problem solving for problem solving's sake; strategies and processes can be emphasized, with less time devoted to the computation within the problem-solving context.

PROBLEM Two friends are bidding each other farewell. They agree to meet each other in exactly 5 years from this moment. To remind themselves of the event, they agree to snap their fingers once every minute until they meet again. How many times will they each snap their fingers until they meet?

Discussion The problem merely requires finding the number of minutes in five years. With a calculator this is simply done by multiplying $5 \times 365 \times 24 \times 60$. Note that there will be either one or two leap years included. This will add 1,440 (or 2,880) "snaps" to the 2,628,000 snaps obtained.

PROBLEM A ball is dropped from a height of 100 feet. Assume that on each bounce it rebounds one-half of the distance of the previous bounce. When the ball stops, what is the total distance through which it traveled?

Discussion This is really a problem in limits. Students are asked to sum the infinite geometric series

$$100 + 50 + 50 + 25 + 25 + 12.5 + 12.5 + \cdots$$

If we isolate the first term, we can deal with half of the series:

$$50 + 25 + 12.5 + 6.25 + 3.125 + \cdots$$

With a calculator, students quickly find that the sum of the series approaches 100. We then double the sum and add the 100. Thus the limit of the series is 300 feet. Notice that the formula for the sum of an infinite geometric progression,

$$S = \frac{a}{1 - r}$$

yields the same result when the series is treated in a similar manner.

PROBLEM A major hamburger chain sold 22 billion hamburgers, each one inch thick. If we stacked these hamburgers, how many miles high would the stack be?

Discussion Since the number 22 billion has 11 digits, it will not fit in most calculator displays. Hence students might multiply 5,280 by 12 to find out how many hamburgers there are in one mile. This number is then divided into 22 billion to get the answer, 347,222.22 miles.

PROBLEM The radius of the earth is approximately 3,987 miles. Find the length of the equator to four significant figures.

Discussion This problem merely requires evaluating the formula $C = 2\pi r$ where r is known. With a calculator, the answer, 25,040 miles is easily obtained. The problem can readily be extended in the following manner:
How far will a satellite 165 miles above the surface of the earth travel in one rotation?

PROBLEM The distance from the pitcher's mound to home plate on a baseball diamond is 60 feet 6 inches. The distance between the bases is 90 feet. How far is it from the pitcher's mound to first base?

Discussion A drawing such as that shown in Figure 3-18 reveals that this problem can be done by applying the Law of Cosines, $a^2 = b^2 + c^2 - 2bc \cos A$.

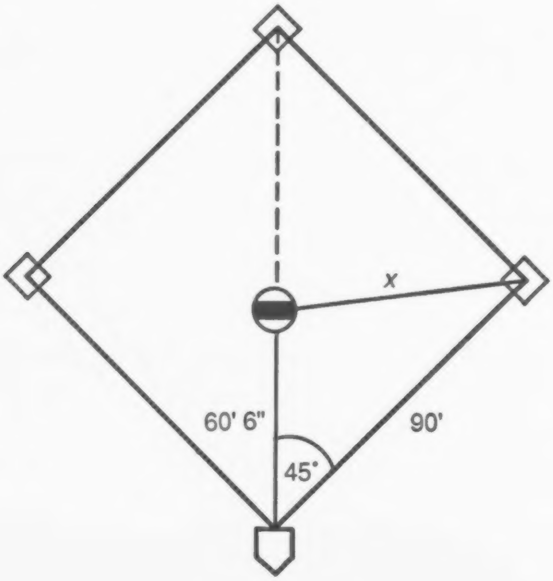


Figure 3-18

$$a^2 = b^2 + c^2 - 2 \cdot b \cdot c \cos A$$
$$x^2 = (90)^2 + (60.5)^2 - 2(90)(60.5)(\cos 45^\circ)$$
$$x^2 = 8100 + 3660.25 - 7699.23$$
$$x^2 = 11{,}760.25 - 7699.23$$
$$x^2 = 4{,}061.02$$
$$x = 63.73$$

The calculator permits us to find the answer quickly.

14. Capitalize on the microcomputer.

The microcomputer is serving students and teachers in many and varied ways, such as providing simulations, computer-aided instruction, tutorials, and word/data processing. Not to be forgotten, however, is the role of programming the computer. Programming provides a direct link between the computer and problem solving. The programming of a computer draws upon many of the skills that are used in problem solving. When students are asked to write a program, they must analyze the task at hand, draw upon their previous knowledge and experiences, and put together an organized plan of steps, operations, and commands that yield the correct results. The format of a program is very much like the heuristics of problem solving.

Although any language can serve, we have written the programs in this section in BASIC for the Apple computer.

PROBLEM Tell what the outcome will be for each of the following programs.

```
(a)  10  LET P = 2 * L + 2 * W
     20  L = 14.6
     30  W = 7.2
     40  PRINT "THE PERIMETER IS "; P
     50  END
```

```
(b)  10  LET A = 1
     20  LET B = 1
     30  PRINT "1"; ", 1 , " ;
     40  LET C = A + B
     50  IF C = 100 THEN 100
     60  PRINT C " , " ;
     70  LET A = B
     80  LET B = C
     90  GOTO 40
    100  END
```

Discussion An activity such as this one provides students with an opportunity to analyze a sequence of steps. This is a skill similar to that which is needed in analyzing a problem. Notice that program (b) will print the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89.

PROBLEM Write a program for finding the area of a rectangle, given its length and width.

Discussion In order to do this, the student must know what a program is, what a rectangle is, and what is meant by length, width, and area. In other words, he or she must understand what is being asked and what is given. This is exactly the same as the first phase of the problem-solving process. The student must now review his or her knowledge of areas and rectangles to find the proper formula. Now this information is synthesized and the program written. This carries the student through the "solve" phase.

```
 5  HOME
10  REM AREA OF A RECTANGLE
20  PRINT "TYPE IN THE LENGTH IN INCHES"
30  INPUT L
35  PRINT
40  PRINT "TYPE IN THE WIDTH IN INCHES"
50  INPUT W
55  PRINT : PRINT
60  LET A = L * W
70  PRINT "THE AREA OF THE RECTANGLE IS "; A ; "SQUARE
     INCHES"
80  END
```

The student now runs the program. Does it work? (This is the "Review" phase.) If not, why not? If it does run correctly, how could it be modified to extend to other geometric figures? (Extend.)

PROBLEM Here is a program designed to find the average of three numbers. Some steps have been omitted. Supply the missing steps, then run the program.

```
10  REM FINDING AVERAGES
20  INPUT A
30
40  INPUT C
50  LET S = A + B + C
60
70  PRINT "THE AVERAGE OF THE THREE NUMBERS IS "; M
80
```

Discussion This program is an exact parallel to the problem-solving situation in which there is missing data. Being able to determine what information is required to resolve a problem situation and to supply such information assures us that the problem solver has a thorough understanding of the problem situation he or she is facing.

PROBLEM Write a program to find the hypotenuse of a right triangle when the two legs are given.

Discussion Here is one possible program:

```
10 HOME
20 INPUT A
30 INPUT B
40 LET W = A * A
50 LET X = B * B
60 LET Y = W + X
70 LET Z = (W + X) ↑ (1/2)
80 PRINT "THE HYPOTENUSE OF THE RIGHT TRIANGLE IS "; Z
90 END
```

PROBLEM Write a program to solve the bouncing ball problem on page 85.

15. Use strategy games in class.

There is evidence to show that students who have been guided in the playing of strategy games have improved their problem-solving ability. This section discusses the use of these strategy games as a vehicle for teaching problem solving. These games do *not* necessarily involve arithmetic skills. Rather, to be considered a strategy game, the game must meet the following conditions:

1. The game must have a definite set of rules for the players.
2. The game must be played by at least two players, each of whom has a goal; these goals must be in conflict with each other.
3. The players must make "intelligent" choices of moves, based upon whatever information is available at the time of the move.

4. Each player tries to stop the opponent from achieving the goal before doing so himself or herself.

Notice that "luck" or "chance" should play a minimal role in strategy gaming.

Games have a strong appeal for children and adults alike. In fact, most people enjoy games. Witness the many books of puzzles and games that are sold in bookstores, as well as the puzzles and game sections that appear on napkins in restaurants or in magazines on airplanes. Currently, an entirely new wave of strategy games utilizing the personal computer is appearing.

Children have been exposed to games and gaming all their lives. They have learned what a game is, that games have rules to be followed, and that it is often possible to win at a particular game consistently by developing a strategy to follow. Most of your students are already familiar with some of the basic strategy games, such as tic-tac-toe, checkers, and chess. They already know some basic strategies for these games.

To students, games are real-world problem situations. They want to win, and they enjoy playing games. Remember that skills acquired under enjoyable conditions are usually retained for longer periods of time than are skills acquired under stress or other adverse conditions.

When we develop a strategy for winning at a strategy game, we usually go through a series of steps that closely parallel those used in a heuristic system for solving problems.

Strategy gaming

1. Read the game rules. Understand the play of the game. What is a "move"? What pieces are used? What does the board look like? What is a "win"? When is a game over?
2. Correlate the rules with those from any related game. Is there a similar game whose strategy you know? Select several possible lines of play and follow them in an attempt to win the game.
3. Carry out your line of play. Can you counter your opponent's moves as the game proceeds?
4. Look back. If your strategy produced a win, will it work every time? Try alternative lines of play, alternative moves.

The similarity between this sequence and the heuristics we suggested in Chapter 2 is marked, indeed.

In order to use strategy games effectively with your students, you need a variety of games. These games can be found in many places. Best of all, games already known to the students can be varied by changing the rules, the pieces, or the game board. In these cases, students need not spend an inordinate amount of time learning all about a "new" game; they can immediately move on to developing a strategy for the play. For example, most of your students already know the game of tic-tac-toe. Under the usual rules, the player who first gets three of his or her own marks (usually X's and O's) in a straight line (vertically, horizontally, or diagonally) is the winner. A simple rule change might be that the first player who gets three of his or her own marks in a straight line is the loser. This creates an entirely new line of play and requires a different strategy.

When you use these games with your students, have them analyze and discuss their play. Have them record their moves in each game. You can help them in this analysis by asking key questions, such as:

1. Is it a good idea to go first?
2. Is it a good idea to play a defensive game (that is, to block your opponent)?
3. If you won, was it luck? Or will your strategy produce a win again?

Have the students play each game several times. Have them refine their strategy each time. Discuss with the class the strategies that consistently lead to a "win."

You can find many examples of strategy games in a local toy store or by looking through the many game and puzzle books available in bookstores. A brief collection of strategy games has been provided in Section A.

SECTION A

A Collection of Strategy Games

TIC-TAC-TOE VARIATIONS

Since the basic game of tic-tac-toe is already known to most students, it becomes a logical place to find a wide assortment of variations.

1. Reverse Tic-Tac-Toe

This game has already been described in Chapter 3. It simply changes the requirements for a "win." Students must take turns placing an X or an O on the board, and try to *avoid* getting three marks in a row.

2. Triangular Tic-Tac-Toe

This game uses the basic rule—that is, the player scoring three of his or her marks in a straight line is the winner. The playing surface, however, has been changed into the triangular array shown in Figure A-1, rather than the usual square array.

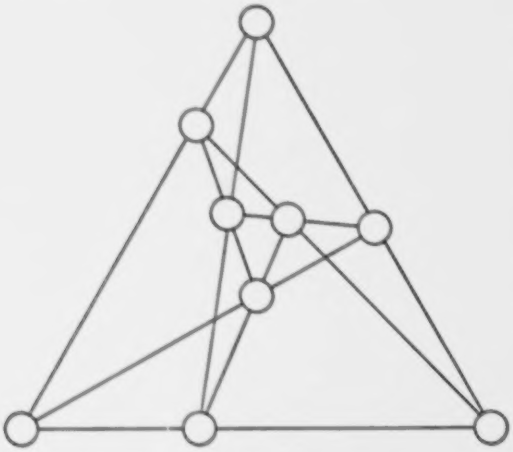


Figure A-1. The Playing Board for Triangular Tic-Tac-Toe

3. Put 'Em Down Tic-Tac-Toe

Instead of making marks on a tic-tac-toe board, each player is provided with three markers or other playing pieces. These are alternately placed on any of the line intersections (circles) on the playing surface in Figure A-2. The center circle may *not* be used by either player as his or her first play. After all the playing pieces have been placed, each player in turn moves one of his or her own pieces along a line to the next vacant cell. The winner is the first player to get three of his or her own pieces in a row (vertically, horizontally, or diagonally).

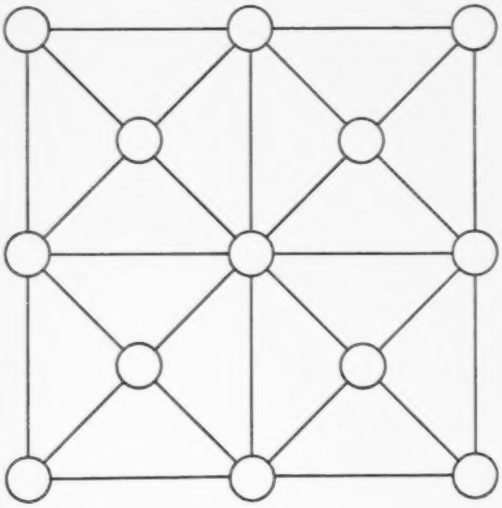


Figure A-2. The Playing Board for Put 'Em Down Tic-Tac-Toe

4. Point Score Tic-Tac-Toe

This version of tic-tac-toe is played on a 25-square playing surface. Players take turns placing a marker of an identifiable color in any unoccupied square on the surface. The object of the game is for a player to get as many pieces of his or her own color as possible in a row (vertically, horizontally, or diagonally). A playing piece is counted in all rows in which it appears. Players score 1 point for three in a row, 2 points for four in a row, 5 points for five in a row. When the square board is completely filled up, players total up their scores to find the winner. The diagram in Figure A-3 shows the playing surface for the game, with a 1-point and 2-point score (horizontally and diagonally, respectively). Notice how one marker counts twice, once in each row.

5. Wild Card Tic-Tac-Toe

This version of tic-tac-toe is played on the traditional 3 × 3 square array commonly used in tic-tac-toe. The variation in this version allows either player to put either mark anywhere on the playing

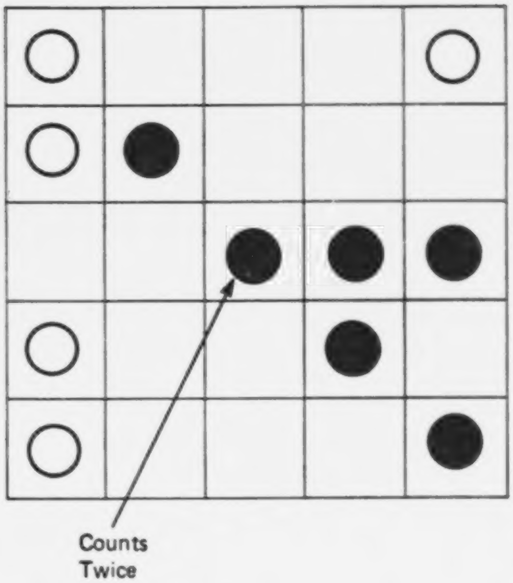


Figure A-3. The Playing Board for Point Score Tic-Tac-Toe

surface when his or her turn comes. (Either an X or an O may be placed by either player at any time during the game.) The winner is the player who *completes* a straight line of three marks of either kind in his or her turn.

6. Three-Person Tic-Tac-Toe

Most versions of tic-tac-toe are games between two players. In this version, however, three people play. Players use either an X, an O, or an I as a marker, and the game is played on a board that contains 6 × 6 or 36 squares. Players put their own mark anywhere on the playing surface in turn. The first player to get three of his or her own marks in a row is the winner.

7. Big 7 Tic-Tac-Toe

This game is played on a playing surface consisting of 49 squares in a 7 × 7 array. Two players take turns placing either an X or an O anywhere on the playing surface. Each places his or her own mark. The first player to get four marks in a row is the winner.

8. Double Trouble Tic-Tac-Toe

This game is played on a 25-square (5 × 5) board. Players take turns placing either two X's or two O's anywhere on the board. The first player to get four of his or her marks in a row (horizontally, vertically, or diagonally) is the winner.

9. As You Wish Tic-Tac-Toe

This game is played on a 25-square (5 × 5) playing surface. Each player in turn may place on the board as many X's or O's as he or she wishes, provided they are all in the same vertical or horizontal row. The player who puts his or her mark into the twenty-fifth or last vacant cell on the board is the winner.

10. Dots-in-a-Row Tic-Tac-Toe

The game is played on a surface as shown in Figure A-4. Players take turns crossing out as many dots as they desire, provided the dots all lie in the same straight line. The player who crosses out the last dot is the winner.

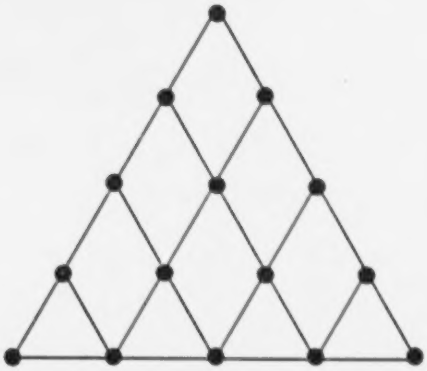


Figure A-4. The Dots-in-a-Row Board

11. Tac-Tic-Toe

This game is played on a 4 × 4 square surface. Each player has four chips or markers of a single color. The starting position is shown in Figure A-5. Players take turns moving a single piece of their own color. A move consists of moving one piece onto a vacant square either horizontally or vertically, but not diagonally. There is no

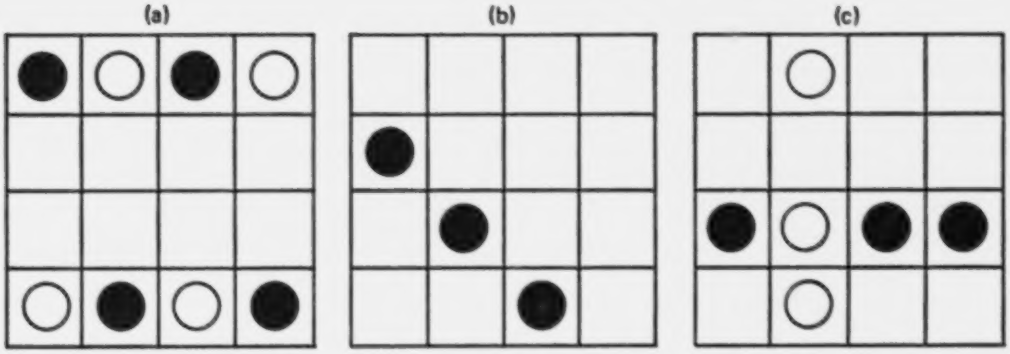


Figure A-5. (a) Starting Position for Tac-Tic-Toe. (b) A Win Position for Tac-Tic-Toe. (c) Nobody Wins in These Positions.

jumping or capturing in this game. No piece can be moved into an already occupied square, but must be moved into an open, adjacent square. The player who moves three of his or her own pieces into a row (horizontally, vertically, or diagonally), with no intervening spaces or intervening squares occupied by an opponent's piece, is the winner.

12. Tac-Tic-Toe, Chinese Version

This game is played on the surface shown in Figure A-6. Players use four chips, each player having a different color. Starting position is as shown in Figure A-6. Each player may move only his or her

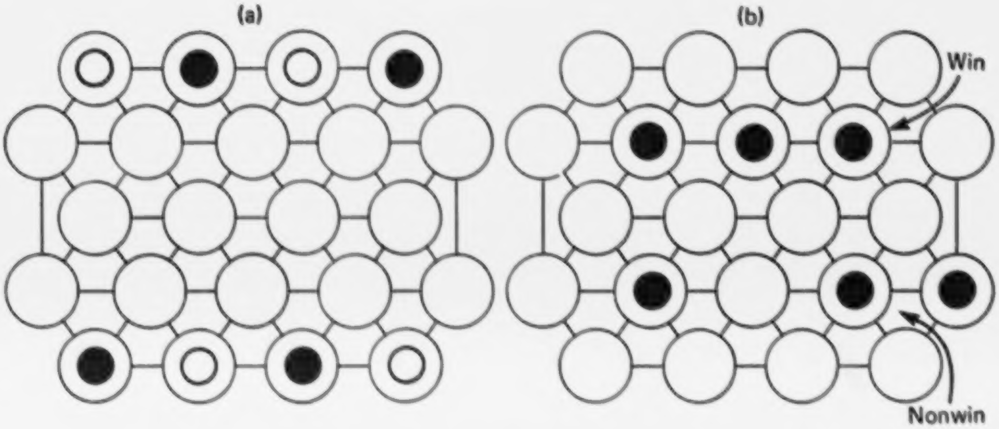
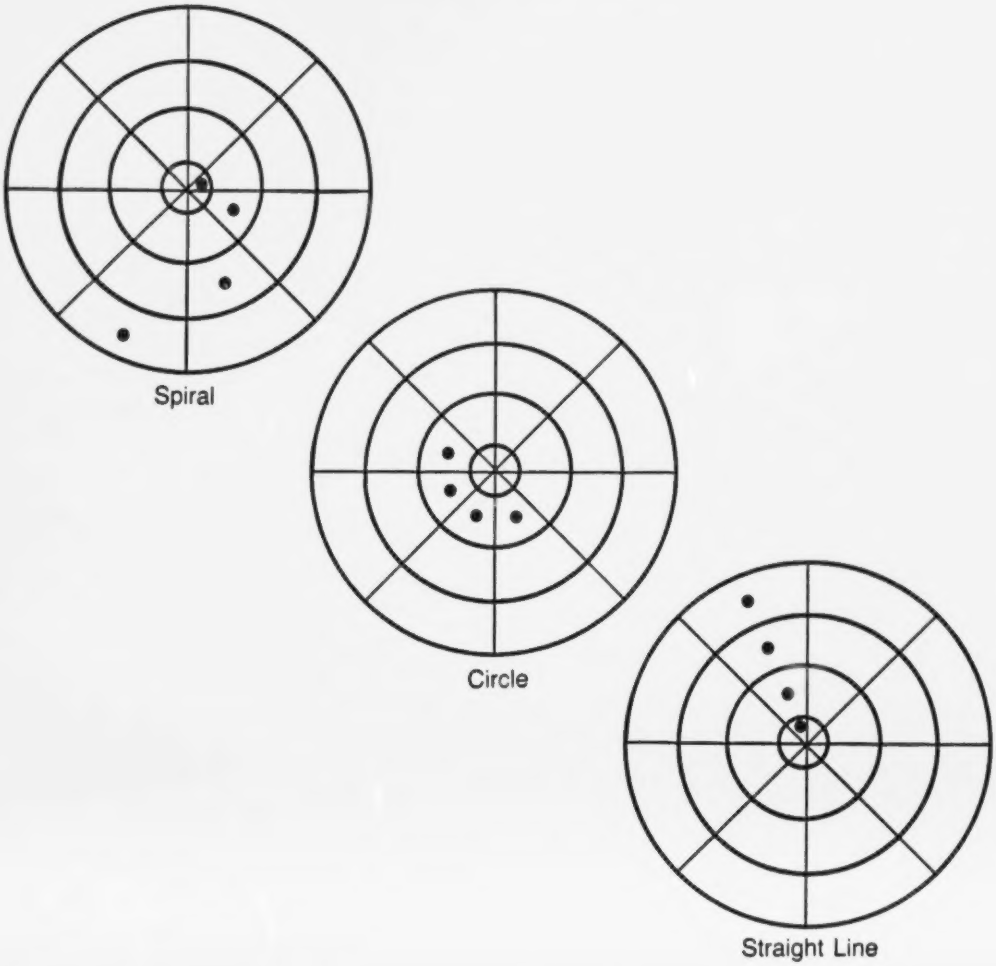


Figure A-6. (a) Starting Position for Tac-Tic-Toe, Chinese Version. (b) A Winning Position and a Nonwinning Position.

own chips. A move involves placing one of one's own pieces into an adjacent, vacant cell, following the lines on the board. There is no jumping and no capturing. The player who places three of his or her own pieces in the same straight line, with no vacant spaces intervening and none of the opponent's pieces intervening, is the winner.

13. Spiral Tic-Tac-Toe

This game is played on the game board shown in Figure A-7. Two players alternate turns placing their mark (an X or an O) in any



Spiral

Circle

Straight Line

Figure A-7

empty spaces on the game board. A player wins when he or she has four in a "row"—either in a straight line, a circle, or a spiral.

BLOCKING STRATEGY GAMES

14. Blockade

Playing pieces are placed on cells A and B in Figure A-8 for player 1, and on cells C and D for player 2. Players take turns moving one playing piece along lines on the playing surface into any vacant, adjacent circle. No jumps or captures are permitted. A player loses when he or she cannot move either of his or her two pieces in his or her turn.

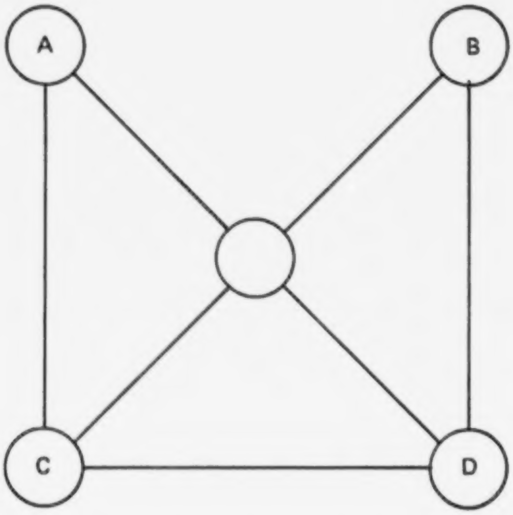


Figure A-8. Playing Surface for Blockade

15. Pawns

This game is played on a 3 × 3 array of squares. Each player has three chips or markers of a single color. Starting position is as shown in Figure A-9. Players take turns moving one of their own pieces. Each piece may be moved one square forward, or one square diagonally if a capture is made. A diagonal move can be made *only* with a capture, and a capture cannot be made on a forward move. A captured piece is removed from the board. A player is a winner when he or she either (1) places one piece in the opponent's starting row or (2) makes the last possible move on the playing board. As a variation, players can play the game on a 4 × 4 or a 5 × 5 board, with an adjusted number of pieces.

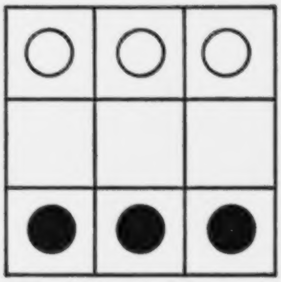


Figure A-9. Starting Position for Pawns

16. Hex

The game of Hex is played on a diamond-shaped board made up of hexagons. (See Figure A-10.) The players take turns placing an X or an O in any hexagon on the board. The winner is the first player to make an unbroken path from one side of the board to the other. Blocking moves and other strategies should be developed as the game proceeds. The corner hexagons can belong to either player.

17. Bi-Squares

The game is played on a playing surface that consists of 16 squares in one long continuous row. Players take turns placing their mark (an X for player 1 and an O for player 2) into each of two adjacent, unoccupied squares on the board. The player who makes the last successful move on the board is the winner.

18. Domino Cover

This game is played on a standard 8 × 8 checkerboard, and uses a set of dominoes that will cover two adjacent squares, either hori-

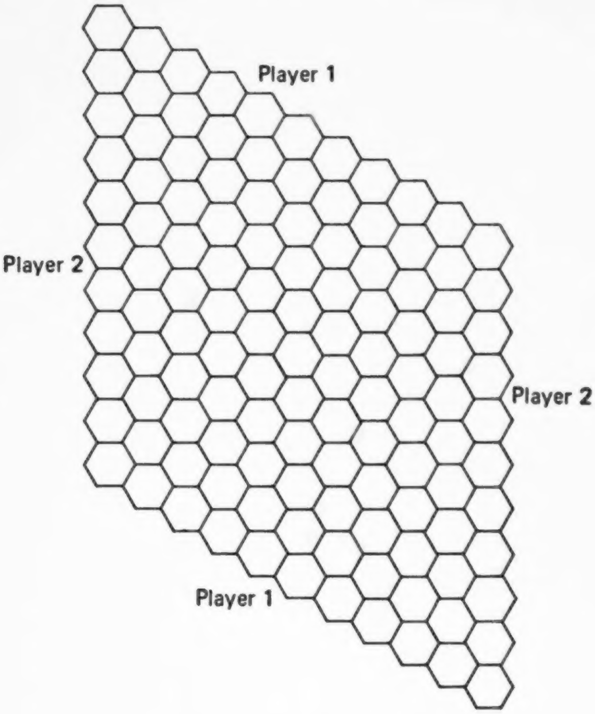


Figure A-10. Hex Board

zontally or vertically. Players take turns placing a domino anywhere on the board, according to the following rules: (1) player 1 can only place dominoes in a horizontal direction; (2) player 2 can only place dominoes in a vertical direction. The loser is the player who cannot make a move by placing a domino in the correct position.

19. Triahex

The game is played on a game board consisting of 6 points that are the vertices of a regular hexagon. (See Figure A-11.) There are exactly 15 lines that can be drawn connecting the 6 points. The game is played by two people, each of whom uses a different colored marker. Players take turns drawing any 1 of the 15 lines. The first player who is forced to finish a triangle formed *completely* with the lines of his or her own color is the loser. Only a triangle whose vertices are points of the original hexagon is to be considered.

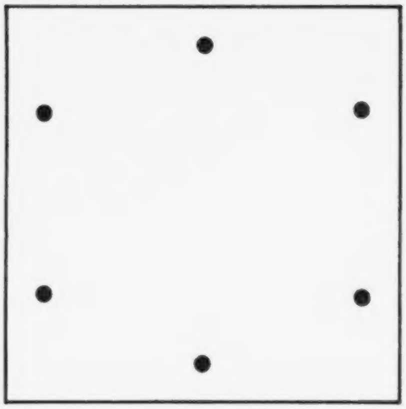
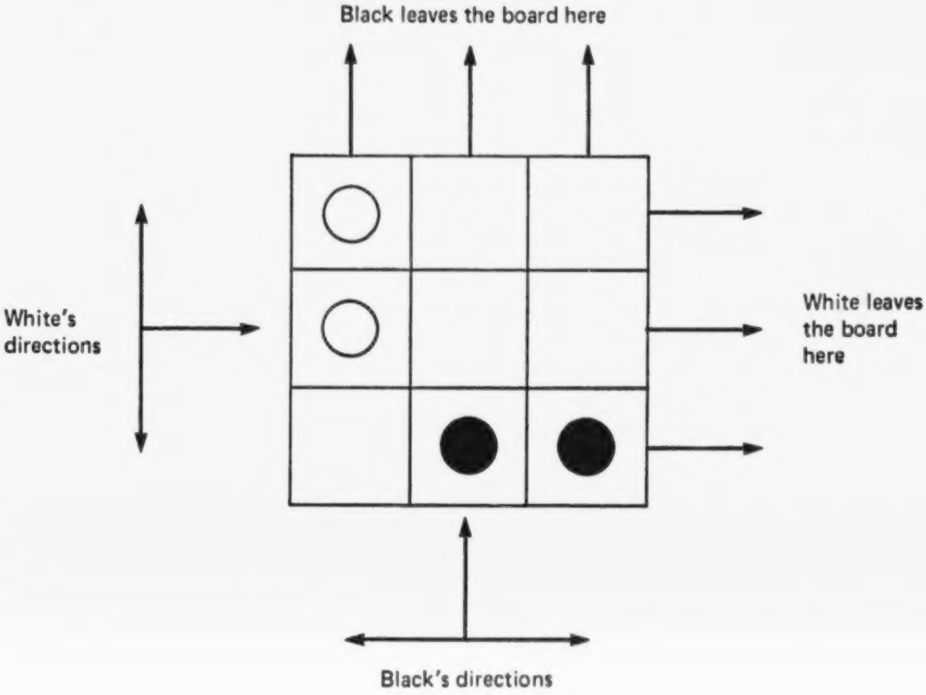


Figure A-11. The Six Vertices of a Regular Hexagon in Triahex

20. Dodge 'Em

The game is played on a 3 × 3 square board with two black checkers and two white ones. Starting position is shown in Figure A-12. Black sits on the south side of the board, white on the west side. Players



Black leaves the board here
White's directions
White leaves the board here
Black's directions

Figure A-12. A Dodge 'Em Board

may move a checker forward or to their left or right, unless the checker is blocked by another counter of either color or the edge of the playing surface. Each player's goal is to move all of his or her pieces off the far side of the board. Black moves north, west, or east and tries to move all of his or her pieces off the board on the north side; white moves east, north, or south and tries to move his or her pieces off the board on the east side. There are no captures or jumps. A player must always leave the opponent a legal move or forfeit the game. The first player to get his or her two pieces off the board is the winner.

21. Tromino Saturation

The game is played on a 5 × 5 square board. The playing pieces consist of the two basic tromino shapes shown in Figure A-13. Each tromino piece should exactly cover three squares on the playing surface. Players take turns placing one of the pieces of either shape anywhere on the playing surface. The first player who cannot place a piece to exactly cover three squares is the loser. (In order to allow each player a full choice of which piece to select on each play, prepare eight pieces of each kind.) If the size of the board is increased to a 6 × 6 square board, prepare twelve of each piece.

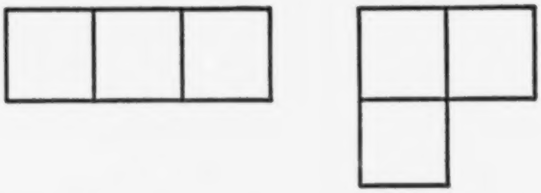


Figure A-13. The Two Basic Tromino Shapes for Saturation

CAPTURE STRATEGY GAMES

22. Solitaire

This a strategy game for one person. The playing surface consists of a board with 15 circles, as shown in Figure A-14. Place chips or other counters on all of the cells except the darkened cell. The player must remove as many counters as possible by jumping counters over adjacent counters (along lines) into empty cells. The jumped counters are removed from the board. All counters but one can be removed in this manner. A winning game is one in which only one counter remains. A variation for experienced players is to try to make the one remaining counter end the game in the darkened cell.

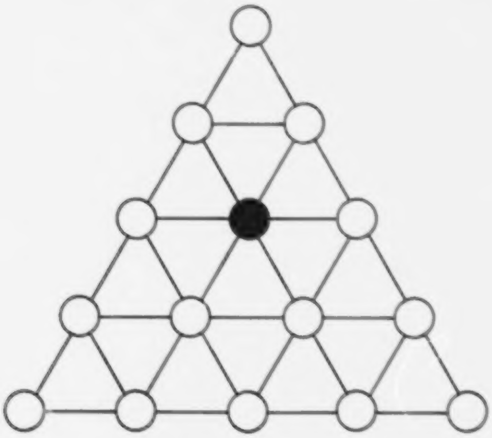


Figure A-14. A Solitaire Board

23. Spot

Spot is played on a circular playing surface, as shown in Figure A-15. Place a penny on spot 2, a dime on spot 15. Players take turns; one moves the penny, the other moves the dime. Moves are made along any solid line into an adjacent spot. The penny moves first. The object of the game is for the penny to "capture" the dime by

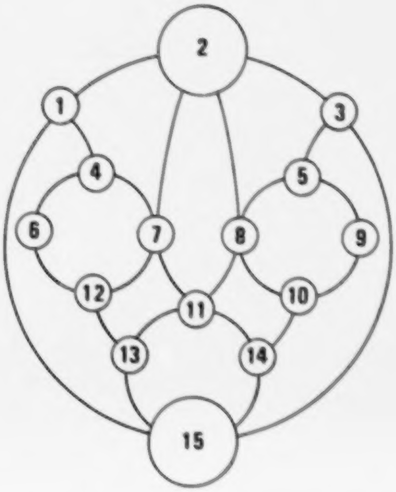


Figure A-15. A Spot Playing Surface

moving into the spot the dime occupies. The capture must be made within six moves. The penny player loses if he or she has failed to catch the dime by the end of his or her sixth move.

24. Fox and Geese

This game, for two players, is played on a surface with 33 cells, as shown in Figure A-16. The fox marker and the thirteen goose markers are placed as shown. The fox can move in any direction along a line—up, down, left, or right. The geese may not move backwards. The fox can capture a goose by making a short jump over a single goose along a line into the next cell, provided that cell is vacant. The fox can make successive jumps on any one turn, provided vacant cells exist. The geese win if they can corner the fox so that he cannot move. The fox wins if he captures enough geese so that they cannot corner him.

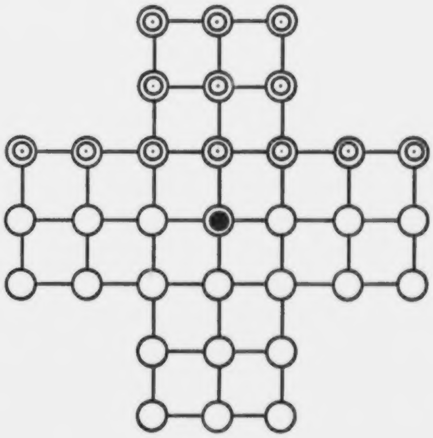


Figure A-16. Starting Position for Fox and Geese

25. The Wolf and the Farmers

One player has a single playing piece, the wolf. The other player has seven farmers. The wolf begins by placing his piece on the top circle of the triangular board in Figure A-17. The second player places one farmer anywhere else on the board. Each time the wolf moves, an additional farmer is placed on the board. All pieces move the same way, one circle at a time, along the marked lines. However, the farmers may not move until all of them have been entered on the board. The farmers may not capture; the wolf captures by jumping over a farmer along a line to a next cell, which must be vacant. The jumped piece is removed from the board. Successive captures are allowed. The wolf wins if he captures enough farmers so that they cannot confine him. The farmers win if they succeed in confining the wolf so that he cannot move.

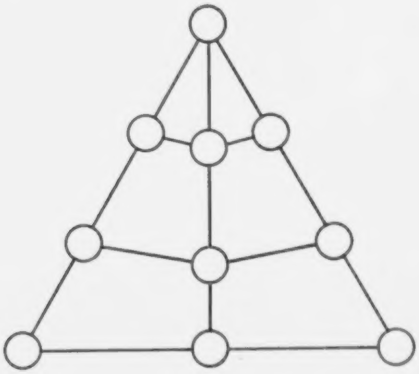


Figure A-17. Playing Surface for the Wolf and the Farmers

26. Sprouts

Three dots are placed in a triangular array on a piece of paper. Players take turns drawing a line connecting any two dots, or connecting a dot to itself. After a line is drawn, a new dot is placed approximately midway between the two dots being connected, along the connecting line. No lines may cross, and no more than three lines may terminate in a single point. The last player to make a successful

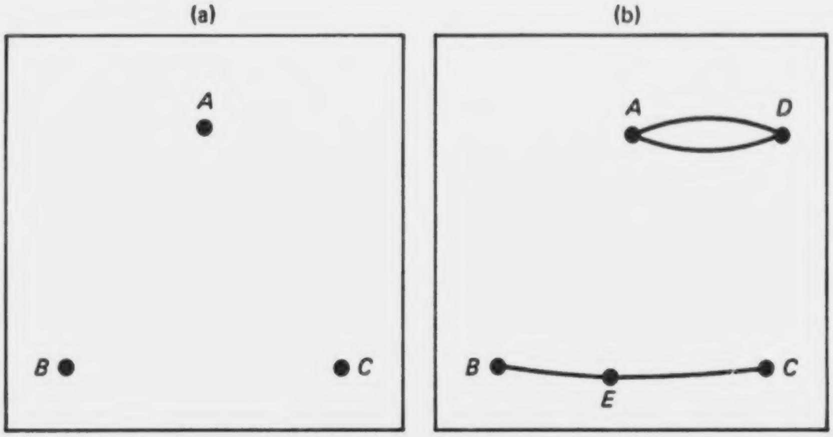


Figure A-18. (a) A Sprouts Board. (b) Typical Moves in Sprouts.

move is the winner. See Figure A-18. The new point *D* is shown along the line connecting point *A* to itself. The new point *E* is shown along the line connecting *B* to *C*. Notice that points *D*, *A*, and *E* each have two lines terminating.

27. Square-Off

The game is played on a checkerboard. Two players take turns placing one checker anywhere on the board. Each player uses checkers of one color. The game continues until the checkers of one color form the vertices of a square of any size. The player using that color is declared the winner.

OTHER STRATEGY GAMES

28. Number Colors

This is a numerical version of the commercial game of "Master Mind." The first player chooses a number made up of 3 different digits. The number is kept hidden. The second player must use logic and strategy to discover the hidden number. The guesses are answered by a color description:

Red	one digit is correct and is in the correct position;
White	one digit is correct but not in the correct position;
Black	no digits are correct.

The responses are not given in any particular order. Thus, a response of Red–White may also be given as White–Red. Here is part of a sample game, with the hidden number 123.

First guess	789	*Response:* Black (no digits are correct)
Second guess	156	*Response:* Red (the digit "1" is correct and in the correct position)
Third guess	135	*Response:* Red–White or White–Red (the "1" is correct and in the correct position; the "3" is correct but not in the correct position)
Fourth guess	132	*Response:* Red–White–White or White–White–Red or White–Red–White

The game continues in a similar manner until the number is correctly guessed. The number of guesses is the player's score. The players then reverse roles and try to guess the number in fewer guesses.

29. Taxman

There are several variations of this well-known game that can be profitably explored by your students. The basic game begins by placing the numbers 1 through 40 on the board. (This sequence can be increased or decreased, depending on the experience of the students). The first player chooses any number in the sequence. This number becomes part of that player's score and is removed from the sequence. The second player then gets all the *factors* of that number for his or her score. These too, are removed from the sequence. Thus, if player #1 selects 20 for a first choice, player #2 gets 1, 2, 4, 5, and 10. It is now player #2's turn to select a number from those remaining in the sequence. Now player #1 gets the factors of this number *that are still in the sequence*. Players continue alternating selection of a number and its factors until all the numbers in the sequence have been taken. The scores are then totaled, and the player with the higher score wins the game. Note that after the selection of a number, it is possible that no factors of that number may remain in the sequence; thus a player may get no factors for a score. Here is an example of the beginning of a possible game:

Round 1	*Score*
Player #1 selects 20.	Player #1: 20
Player #2 gets 1, 2, 4, 5 and 10.	Player #2: 22
Round 2	
Player #2 selects 30.	Player #2: 22 + 30 = 52
Player #1 gets only 3, 6, and 15 (since 1, 2, 5, and 10 are no longer in the sequence).	Player #1: 20 + 24 = 44
Round 3	
Player #1 selects 25.	Player #1: 44 + 25 = 69
Player #2 gets 0 (since 1 and 5 are gone from the sequence).	Player #2: 52 + 0 = 52
Round 4	
Player #2 selects 37.	Player #2: 52 + 37 = 89
Player #1 gets no score.	Player #1: 69 + 0 = 69

Variations

1. One variation of the game can be achieved by a simple rule change. A number may be selected only if at least one of its factors still remains in the sequence. Thus, if player #1 chooses 37, player #2 gets 1. Now neither player can select a prime number, since there is no number (1) remaining for the opponent. This game concludes when neither player can remove a number and leave a factor. At this point, the numbers remaining in the sequence are disregarded for purposes of scoring.

2. The game can be modified so that each player, in turn, plays against the Taxman. Thus player #1 chooses a number and the Taxman gets all the factors of that number. Now player #1 goes again and selects another number. The Taxman again gets all the factors of this number. Play ends when only numbers with no factors remain in the sequence. These numbers are added to the Taxman's score. Player #1's score is the net, or difference, between the player's score and that of the Taxman. Player #2 now tries to gain a better net score. Watching an opponent play can help a student develop a winning strategy for the game.

SOME COMMERCIAL STRATEGY GAMES

30. Basis

A strategy game in which players form numerals in different bases, while preventing their opponents from doing the same. (Holt, Rinehart and Winston Company.)

31. Battleship

A game of strategy in which two players try to sink each other's ships, which are hidden from view. A good introduction to coordinates. (Creative Publications.)

32. Bee Line

Players use strategy while attempting to make a "beeline" across the playing board. (SEE Corporation.)

33. Block 'N Score

A strategy game in which two players work with binary notation. (Creative Publications.)

34. Equations

A game designed to give students practice in abstract reasoning, to increase speed and accuracy in computing, and to teach some of the basic concepts of mathematics. The game can be varied to work in different bases. (Wff 'N Proof.)

35. Foo (Fundamental Order of Operations)

A strategy game in which players try to combine seven cards into any multiple of 12. Extra cards are drawn and discarded until one player calls "Foo!" (Cuisenaire, Inc.)

36. Helix

Another three-dimensional tic-tac-toe game. Players place different colored beads on a series of pins, trying to get four in a row. The pins are not only in straight lines, but also along arcs designated on the playing surface. (Creative Publications.)

37. Kalah

A strategy game involving counting, skill, and logic. Chance is a minimal factor. (Creative Publications.)

38. Mastermind

A secret code of colored pegs is set up out of sight of one player. He or she then has ten chances to duplicate the colors and exact positions of the code pegs. Pure logic! (Cadaco, Invicta, Creative Publications.)

39. Numble

A game similar to a crossword puzzle. Players place tiles with numerals from 0 to 9 on them to form addition, subtraction, multiplication, and division problems. (Math Media, Inc.)

40. On Sets

A series of games that teach some of the basic ideas of set theory and set language. Comes with an excellent manual. Involves union, intersection, complement, etc. (Wff 'N Proof.)

41. Pressups

A player must guide the direction of play so as to press down pegs of his or her own color. Traps may be set. The winner is the player with more of his or her own pegs depressed. (Invicta.)

42. Qubic

Qubic expands tic-tac-toe into a four-level space game. Players win by setting four markers in a straight line in one or several planes. (Parker Brothers.)

43. Racko

By drawing from the pile players attempt to replace cards in their racks so that the numbers read from high to low in numerical sequence. (Milton Bradley Company.)

44. Score Four

Similar to Qubic and Helix. Players place wooden beads on metal pins and need four in a row to score. (Lakeside Toys.)

45. Soma Cube

An elegant cube with irregular sets of combinations of cubes. There are 1,105,920 mathematically different ways to come up with the 240 ways the seven SOMA pieces fit together to form the original cube. (Parker Brothers.)

46. Tromino

Triangular wooden pieces extend the skills and principles needed for dominoes. A great deal of strategy is used, as well as mathematical knowledge. (Creative Publications.)

47. Tuf

A game designed to help students utilize the basic operations of mathematics. Students form equations using tiles with various numerals and operations on them. Both speed and accuracy count. (Avalon Hill Company.)

48. Twixt

A board strategy game with moves and countermoves. Players try to connect a chain of linked pegs before their opponents can do the same. (3-M Company.)

49. Vectors

A game in which thought and ability are used to deceive opponents. The game utilizes a chess-like board with a single piece and cards. (Cuisenaire, Inc.)

50. Wff 'N Proof

The game is based on symbolic logic. Symbols are used to form logical sequences. (Wff 'N Proof.)

SECTION B

A Collection of Non-Routine Problems

A Collection of Non-Routine Problems

The following set of problems has been chosen to provide practice in problem solving for your students. Remember, a problem is merely the vehicle for teaching and learning the problem-solving process. Many of these problems can easily be solved by making use of the power of algebra, but it would be advantageous to have the students discover alternate solutions wherever possible. In all cases, the solutions should be examined and discussed with the entire class. You should keep in mind that the answer is only a part of the solution.

We have attempted to arrange the problems in increasing order of student maturity. However, the actual choice of problems for a particular student or group of students must be made by the classroom teacher. Only he or she is in a position to determine the appropriateness of a given problem for an individual student.

PROBLEM 1 All my pets are dogs except two; all my pets are cats except two; all my pets are parakeets except two. How many cats do I have?

Discussion This is a problem that requires logical thought. Since there are three different kinds of animals, students might begin with one of each kind and work from there. They should arrive at an answer of one dog, one cat, and one parakeet. Thus the answer is one cat.

PROBLEM 2 On a camping trip, Jack wanted to make three slices of toast by placing the bread on the griddle for 30 seconds and then turning each slice over for 30 seconds on the other side. The only thing is, the griddle holds two slices of bread at a time. How can he toast three slices of bread on both sides in $1\frac{1}{2}$ minutes?

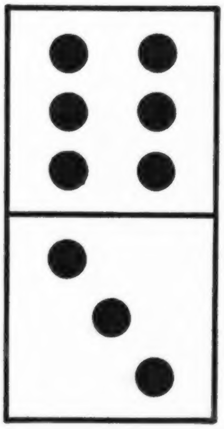
Discussion For Jack to toast

A_1 and B_1	takes 30 seconds
A_2 and C_1	takes 30 seconds
B_2 and C_2	takes 30 seconds

PROBLEM 3 A man takes a 5,000-mile trip in his car. He rotates his tires (4 on the car and 1 spare) so that at the end of the trip each tire has been used for the same number of miles. How many miles were driven on each tire?

Discussion Since four tires are always in use, we are really talking about 4 × 5,000, or 20,000 miles of tire use. Thus, 20,000 divided by 5 tires means each was used for a total of 4,000 miles.

PROBLEM 4 In a domino set, each domino contains two numbers as shown in Figure B-1. Each number from 0 through 6 is paired exactly once with every other number from 0 through 6, including itself. How many dominoes are in a set?



A Sample Domino

Figure B-1

Discussion There are:

7 dominoes of 0 combined with 0 through 6
6 dominoes of 1 combined with 1 through 6
5 dominoes of 2 combined with 2 through 6
4 dominoes of 3 combined with 3 through 6
3 dominoes of 4 combined with 4 through 6
2 dominoes of 5 combined with 5 through 6
1 domino of 6 combined with 6
28

There are 28 dominoes in a set.

PROBLEM 5 In a certain family, a boy has as many sisters as he has brothers. However, each sister has only one-half as many sisters as brothers. How many brothers and sisters are there in this family?

Discussion By Guess and Test, we find that there are 4 brothers and 3 sisters in the family.

PROBLEM 6 A merchant who was liquidating his stock offered various items at the following prices:

7 items at $10 per item
12 items at $8 per item
15 items at $6 per item
6 items at $5 per item

If he sold one-half of this stock on the first day, what is the most money he could have received?

Discussion Since there are a total of 40 items and he sold one-half, he must have sold 20 items. Selecting the 20 items beginning with the most expensive, we get:

$$\begin{array}{r}
\text{7 items at \$10 each} = \$70 \\
\text{12 items at \$8 each} = \$96 \\
\text{1 item at \$6} = \underline{\$6} \\
\$172
\end{array}$$

PROBLEM 7 Joe and Rhoda bought some items in the local pharmacy. All the items they bought cost the same amount, and they bought as many items as the number of cents in the cost of one single item. If Joe and Rhoda spent exactly $6.25, how many items did they buy?

Discussion The first time students attempt this problem, they probably will focus on the fact that all of the items cost the same amount and they spent $6.25. On the next reading, they should realize that Joe and Rhoda bought as many items as the cost of one item. That is, if the cost of the item was five cents, they bought five items. Finally, the students should realize that the problem concerns the square root of 6.25.

PROBLEM 8 A penny weighs approximately 3 grams, while a nickel weighs approximately 5 grams. About how much more does $5.00 in pennies weigh than $5.00 in nickels?

Discussion $5.00 in pennies will be 500 pennies that weigh 1500 grams. $5.00 in nickels will be 100 nickels that weigh 500 grams. The difference will be 1000 grams.

PROBLEM 9 A soccer ball consists of 32 panels, of which 12 are regular pentagons and 20 are regular hexagons. All of the sides are the same length. A seam exists wherever 2 panels share a common edge. How many seams are found on a soccer ball?

Discussion There are 5×12 or 60 pentagonal edges and 20×6 or 120 hexagonal edges, a total of 180 edges. Since two edges create a seam, there must be 90 seams.

PROBLEM 10 A man has 169 horses to be shared among his three daughters in the ratio of one-half to one-third to one-fourth. How many horses should each daughter receive?

Discussion Since $\frac{1}{2} = \frac{6}{12}$, $\frac{1}{3} = \frac{4}{12}$, and $\frac{1}{4} = \frac{3}{12}$, the ratio is the same as 6:4:3. Thus the first daughter receives $\frac{6}{13}$ of 169, or 78 horses. The second daughter receives $\frac{4}{13}$ of 169, or 52 horses, while the third daughter receives $\frac{3}{13}$ of 169, or 39 horses.

PROBLEM 11 How many rectangles are there with an area of 94 square inches if the lengths of the sides must be integers?

Discussion Let x represent the length of the rectangle
y represent the width of the rectangle.
Then $xy = 94$ and $x = 94/y$.
Since x is an integer, y must be a factor of 94. But 47 is a prime; therefore, the only possible answers are 94×1 and 47×2. There are only two possible rectangles.

PROBLEM 12 A woman bought some 15¢ stamps and some 8¢ stamps at the post office. She paid exactly one dollar. How many of each type of stamp could she have bought?

Discussion If we approach this problem algebraically, we arrive at the Diophantine equation $15x + 8y = 100$, where x and y are positive integers. There is only one set of integers that satisfies the equation, namely 4 stamps at 15¢ and 5 stamps at 8¢.

PROBLEM 13 A grocer has three pails: an empty pail that holds 5 liters, an empty pail that holds 3 liters, and an 8-liter pail that is filled with apple cider. Show how the grocer can measure exactly 4 liters of apple cider with the help of the 5-liter and 3-liter pails.

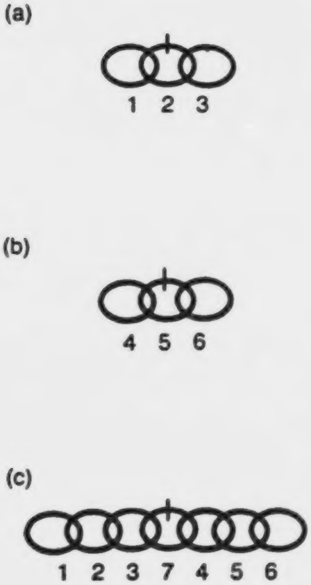
Discussion Make a chart to show how the grocer might do the pourings:

3-liter pail	*5-liter pail*	*8-liter pail*
0	0	8
0	5	3
3	2	3
3	0	5
0	3	5
3	3	2
1	5	2
1	0	7
0	1	7
3	1	4

While this *does* yield an answer to the problem, notice that this answer is not a minimum number of pourings (which was not called for in the original problem). For example, the first three steps can be ignored; one could begin directly with step 4 (3 liters in the 3-liter pail, 5 liters in the 5-liter pail). Students should be encouraged to go on to find the minimum number of pourings needed.

PROBLEM 14 It costs a dime to cut and weld a chain-link. What is the minimum number of cuts needed to make a single chain from seven individual links?

Discussion Make a series of drawings as shown in Figure B-2.



opening link #2 connects 1 - 2 - 3

opening link #5 connects 4 - 5 - 6

opening link #7 connects 1-2-3 to 4-5-6

Figure B-2

PROBLEM 15 If exactly 3 darts hit the target in Figure B-3, how many different scores are possible?

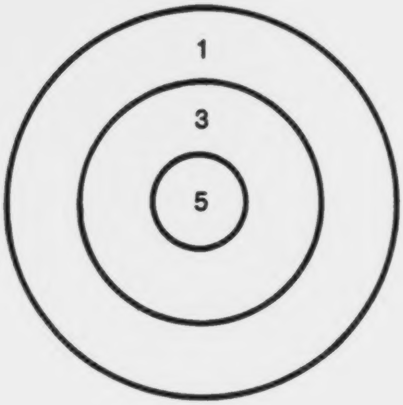


Figure B-3

Discussion Make an exhaustive list:

5's	*3's*	*1's*	*Score*
3	0	0	15
2	1	0	13
2	0	1	11
1	2	0	11
1	1	1	9
1	0	2	7
0	3	0	9
0	2	1	7
0	1	2	5
0	0	3	3

Although there are 10 possibilities, there are two scores of 9, two scores of 7, and two scores of 11. Thus, only seven different scores are possible.

PROBLEM 16 How many triangles are there in Figure B-4?

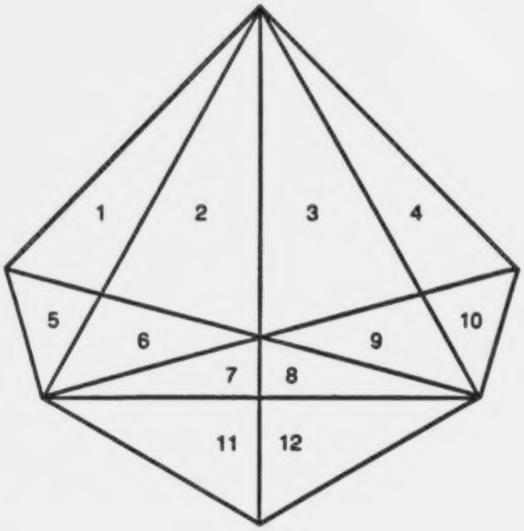


Figure B-4

Discussion An organized list would seem appropriate:

Section B

One region	*Two regions*	*Three regions*	*Four regions*	*Six regions*
1	1, 2	2, 3, 6	1, 2, 3, 9	2, 3, 6, 7, 8, 9
2	1, 5	2, 3, 9	2, 3, 4, 6	
3	2, 6	2, 6, 7	2, 6, 7, 11	
4	3, 4	3, 8, 9	3, 8, 9, 12	
5	3, 9	6, 7, 8	5, 6, 7, 8	
6	4, 10	7, 8, 9	7, 8, 9, 10	
7	5, 6			
8	7, 8			
9	7, 11			
10	8, 12			
11	9, 10			
12	11, 12			

PROBLEM 17

(a) What is the smallest number, the sum of whose digits is 37?

(b) What is the largest number, the sum of whose digits is 37?

Discussion

(a) Since we want the number to be as small as possible, we will keep the number of digits to a minimum. Thus we use as many 9's as possible. The number is 19,999.

(b) To make the number as large as possible, we want to use as many digits as we can. Thus, we use as many 1's as possible. The number is 111 . . . 111 (37 digits, all 1's).

PROBLEM 18

Della saves boxes. She puts them inside of each other to save room. She has 5 large boxes. Inside each of these, she puts 2 middle-sized boxes. Inside each of the middle-sized boxes, she puts 5 tiny boxes. How many boxes does Della have in all?

Discussion

Work the problem using one large box, and multiply by 5. In a large box, there are 2 smaller boxes, each containing 5 tiny boxes. Thus there are 13 boxes. 13 × 5 = 65 boxes in all.

PROBLEM 19

Karen has to number the 396 pages in her biology notebook. How many digits will she have to write?

Discussion

Pages	*Numbers*	*Digits*	*Total*
1–9	9	9	9
10–99	90	180	189
100–396	297	891	1080

She will write 1,080 digits.

PROBLEM 20

(a) If a pile of 100 sheets of paper is 1 centimeter high, how high is a pile of 1 million sheets of the paper?
(b) If a pile of 100 sheets of paper is 1 centimeter high, how many sheets of the paper would be needed to make a pile that is 1 mile high?

Discussion

(a) Using a calculator, we arrive at 100 meters, or approximately 328 feet.
(b) Again using a calculator, we arrive at 16 million sheets.

PROBLEM 21

A telephone company installs and maintains telephone systems for small businesses. They charge a flat rate of $6 a month for each telephone. After a cost survey, they find that to install and maintain telephones on a monthly basis costs them as follows: $3 for 1 telephone; $5 for 2 telephones, $8 for 3 telephones; $12 for 4 telephones, and so on. If the pattern continues, how many telephones can they install before they lose money?

Discussion

The problem can be done expressing the information in a table:

Number of telephones	1	2	3	4	5	6	7	8	9	10	11
Charge/mo.	6	12	18	24	30	36	42	48	54	60	66
Cost/mo.	3	5	8	12	17	23	30	38	47	57	68
Net profit	3	7	10	12	13	13	12	10	7	3	−2

They lose money on the eleventh telephone.

If we express the net profit as an algebraic equation in terms of p (the number of telephones),

$$\text{N.P.} = 6p - \left(\frac{1}{2}p^2 + \frac{1}{2}p + 2\right)$$

(This is arrived at by the use of finite differences.)

$$\begin{aligned}\text{N.P.} &= 6p - \frac{1}{2}p^2 - \frac{1}{2}p - 2\\ &= -\frac{1}{2}p^2 + \frac{11}{2}p - 2\end{aligned}$$

Setting this equal to 0 (the "break even point"), we obtain $p = 0$ or $p = 10.6$. Thus the company loses money on the eleventh telephone.

PROBLEM 22 Jeff and Mike breed tropical fish. Mike decides to abandon the hobby and give his fish away. First he gives half of them and half a fish more to Scott. Then he gives half of what is left and half a fish more to Billy. That leaves Mike with only 1 fish, which he gives to Jeff. How many fish did Mike start with?

Discussion Although we can solve this problem algebraically, the equation formed is rather complicated:

$$x - \left(\frac{x}{2} + \frac{1}{2}\right) - \left[\frac{x - \left(\frac{x}{2} + \frac{1}{2}\right)}{2} + \frac{1}{2}\right] = 1$$

If we work backwards (and assume that Mike doesn't cut any fish in half), the students should come to realize that they are working with a set of odd numbers. Since only 1 fish remains after Mike's last gift, he must have had 3 fish before giving the gift to Billy (half of 3 fish is 1½ fish + ½ fish = 2 fish). Working backwards in a similar manner, we find that Mike must have started with 7 fish and given 3½ + ½ or 4 fish to Scott.

PROBLEM 23 If a brick balances with three-quarters of a brick plus three-quarters of a pound, then how much does the brick weigh?

Discussion The three-quarters of a pound weight must exactly balance the missing one-fourth of the brick. Thus ¼ of a brick weighs $\frac{3}{4}$ of a pound, and $\frac{4}{4}$ of a brick weighs 4 times $\frac{3}{4}$ of a pound, or 3 pounds.

PROBLEM 24 To hang a picture on a bulletin board, Jim uses 4 thumbtacks, one in each corner. If he overlaps the corners, Jim can hang two pictures with only 6 tacks. What is the minimum number of tacks Jim needs to hang 6 pictures?

Discussion A drawing will reveal that two rows of three pictures each is the most efficient way to hang the six pictures. It will only take 12 thumbtacks to do the job.

PROBLEM 25 In the outer reaches of space, there are eleven relay stations for the Intergalactic Space Ship Line. There are space ship routes between the relay stations as shown in the map in Figure B-5. Eleven astronauts have been engaged as communications operators, one for each station. The people are Alex, Barbara, Cindy, Donna, Elvis, Frances, Gloria, Hal, Irene, Johnny, and Karl. The two people in stations with connecting routes will be talking to each other a great deal, to discuss space ships that fly from station to station. It would be helpful if these people were friendly

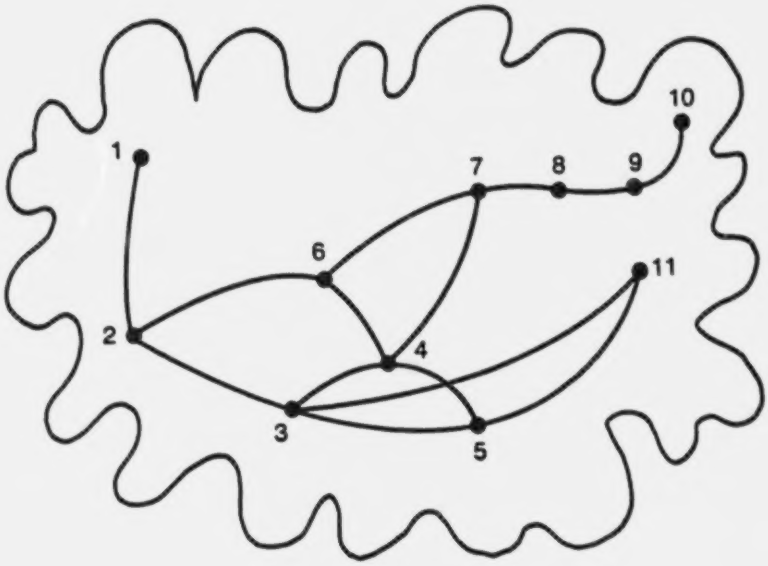


Figure B-5

with each other. Here are the pairs of people who are friends:

Alex–Barbara
Gloria–Johnny
Donna–Irene
Cindy–Hal
Johnny–Cindy
Hal–Frances
Gloria–Irene
Alex–Gloria
Alex–Donna
Donna–Karl
Irene–Karl
Donna–Elvis
Karl–Elvis
Johnny–Irene

Place the eleven people in the eleven stations so that the people on connecting stations are friends.

Discussion An examination of the number of connections for each space station shows the number of "friends" the person assigned to the station must have. Prepare a table showing how many friends each person has. Finding the corresponding numbers in the table will help with the people assignments.

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*
People to whom they talk	B	A	H	I	D	H	J	F	K	I	E
	G		J	E	K		I	C	D	C	I
	D			K			A		G	G	D
				A					J		
Total	3	1	2	4	2	1	3	2	4	3	3

Thus, the person assigned to station 1 must be friendly with the one assigned to station 2. Since this is his or her only speaking friend, the spot must go to either Barbara or Frances. But the person in station 2 must have 3 friends; thus it must be Barbara, whose friend Alex has 3 friends. Continue in a similar manner to locate all 11 people.

PROBLEM 26 A textbook is opened at random. To what pages is it opened if the product of the facing page numbers is 3,192?

Discussion Students can try pairs of successive numbers on their calculators until they find the product 3,192. One alternative method is to take the square root of 3,192. This gives 56.497787. Students then try the integers surrounding this number, namely 56 and 57. An algebraic solution can be formulated with x and $x + 1$ representing the two page numbers:

$$x(x + 1) = 3192$$
$$x^2 + x = 3192$$
$$x^2 + x - 3192 = 0$$

and again students must find two successive integers whose product is 3,192.

PROBLEM 27 Two elevators each leave the sixth floor of a building at exactly 3:00 P.M. The first elevator takes 1 minute between floors, while the second elevator takes 2 minutes between floors. However, whichever elevator arrives at a given floor first must wait 3 minutes before leaving. Which elevator arrives at the ground floor first?

Discussion A simple table will lead to an answer:

		Elevator #1	*Elevator #2*
6		3:00	3:00
5	Arrives	3:01	3:02
	Leaves	3:04	3:02
4	Arrives	3:05	3:04
	Leaves	3:05	3:07
3	Arrives	3:06	3:09
	Leaves	3:09	3:09
2	Arrives	3:10	3:11
	Leaves	3:13	3:11
1	Arrives	3:14	3:13

The second elevator arrives at the first floor one minute ahead of the first elevator.

PROBLEM 28 You have 29 cubes. You make four piles so that the first pile contains 3 more cubes than the second pile; the second pile contains 1 cube less than the third pile; and the fourth pile contains twice as many cubes as the second pile. How many cubes are in each pile?

Discussion Algebraically, students can let x represent the number of cubes in the second pile. Then

First pile $= x + 3$
Second pile $= x$
Third pile $= x + 1$
Fourth pile $= 2x$

Then we form the equation

$$(x + 3) + (x) + (x + 1) + (2x) = 29$$

This is also an excellent problem to be done by an experiment. Provide the students with a set of 29 cubes. By moving them about, the students arrive at the answer 8, 5, 6, and 10.

PROBLEM 29 The following advertisement appeared in the real estate section of a local newspaper:

INVEST FOR THE FUTURE!
LAND FOR SALE–
ONLY 5¢ PER SQUARE INCH!!

Invest your money now in land,
in an area that is soon to be developed.
For information, call or write

Land Developers, Inc.

If you were to buy an acre of land as advertised, what amount would you be required to pay?

Discussion In addition to teaching "number explosions," this problem introduces the topic of advertising and come-on programs. Mathematically, it is an interesting problem for use with the calculator. The only information required is that an acre is 200′ × 200′. Thus,

$$(200 \times 12)(200 \times 12)(.05) = \$288,000 \text{ an acre}$$

PROBLEM 30 On a cheese pizza box, the directions read, "Spread the dough to the edges of a 10″ × 14″ rectangular pan." If you want to make a round pizza with the same thickness, how big a pizza could you make?

Discussion In this problem, the area of the circle must equal the area of the rectangle. Thus,

$$\pi r^2 = L \times W$$
$$\pi r^2 = 10 \times 14$$
$$\pi r^2 = 140$$
$$r^2 = 140/\pi \approx 140/3.14 \approx 44.59$$
$$r = 6.67 \text{ inches (approximately)}$$

The pizza would have a radius of approximately 6.67 inches.

PROBLEM 31 A rectangle has an area of 48 square inches, and its side lengths are integers. What is the smallest perimeter possible?

Discussion Make a table to show the length, width, and perimeter of rectangles whose area is 48.

L	W	P
48	1	98
24	2	52
16	3	38
12	4	32
8	6	28

The smallest possible perimeter would be 28 inches.

PROBLEM 32 Two bicycle riders, Jeff and Nancy, are 25 miles apart, riding towards each other at 15 miles per hour and 10 miles per hour respectively. A fly starts from Jeff and flies toward Nancy and then back to Jeff again and so on. The fly continues flying back and forth at a constant rate of 40 miles per hour until the bicycle riders "collide" and crush the fly. How far has the fly traveled?

Discussion If we try to solve the problem using distance as our main focus, we arrive at a very complex series of equations.

However, carefully examine the time it takes Jeff and Nancy to meet and crush the fly. Since they are moving toward each other at a constant rate of 25 miles per hour, it will only take them 1 hour of bicycling to reach each other. Thus the fly is moving back and forth for one hour, or a total of 40 miles.

PROBLEM 33 Find the value of x if the number $12xxx3$ is a multiple of 11.

Discussion For a number to be divisible by 11, the sum of the "odd" digits must equal the sum of the "even" digits. Thus,

$$1 + x + x = 2 + x + 3$$
$$2x + 1 = 5 + x$$
$$x = 4$$

We check the results, and see that 124,443 is exactly divisible by 11.

PROBLEM 34 Two gentlemen were trying to decide when to open their store for business. "When the day after tomorrow is yesterday," said Marv, "then 'today' will be as far from Sunday as that day which was 'today' when the day before yesterday was tomorrow." On which day of the week were the two men talking?

Discussion Diagramming the conversation will lead us to discover that the men were talking on Sunday.

PROBLEM 35 A square piece of paper is folded in half and cut along the dotted line (the fold) as shown in Figure B-6, to form 2

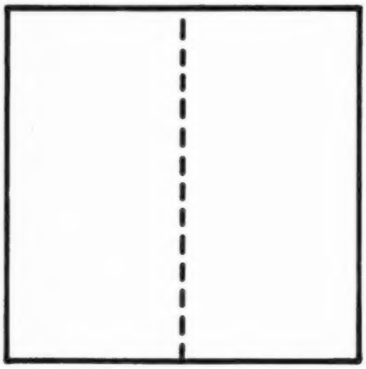


Figure B-6

congruent rectangles. If the perimeter of each rectangle is 30 inches, what was the perimeter of the original piece of paper?

Discussion Let each side of the original square piece of paper be represented by $2s$. Then the perimeter of one of the rectangles is $s + s + 2s + 2s = 6s = 30$, or $s = 5$. Thus, the original square had a perimeter of $8s$, or 40 inches.

PROBLEM 36 The sweep second hand of a quartz watch pulsates 1 beat per second. On July 18th at midnight, I put a new battery in and started the watch. On what day will the second hand register its one-millionth beat?

Discussion Encourage the use of a calculator:

1 beat per second
60 beats per minute
3600 beats per hour
86,400 beats per day

This means there will be 11.57 days in one million beats. The millionth beat will be on July 30th.

PROBLEM 37 A store was selling toothbrushes for 50¢ each. When the price was reduced, the remaining stock sold for $31.93. What was the reduced price of a toothbrush?

Discussion The number of brushes and the price per brush are the factors of 3,193. The only factors of 3,193 are 31 and 103, both primes. Thus 103 toothbrushes were sold for 31¢ each. (How about 3,193 toothbrushes at 1¢ each?)

PROBLEM 38 A family tree for a male bee is very unusual. A male bee has only one parent (a mother), while a female bee has two parents (a mother and a father). How many ancestors does a single male bee have, if we go back for 6 generations?

Discussion We build a tree diagram showing the ancestors of a single male bee, as in Figure B-7.
Notice that the numbers in each generation form a Fibonacci sequence: 1, 2, 3, 5, 8, 13. The single male bee will have 32 "ancestors" if we go back for 6 generations.

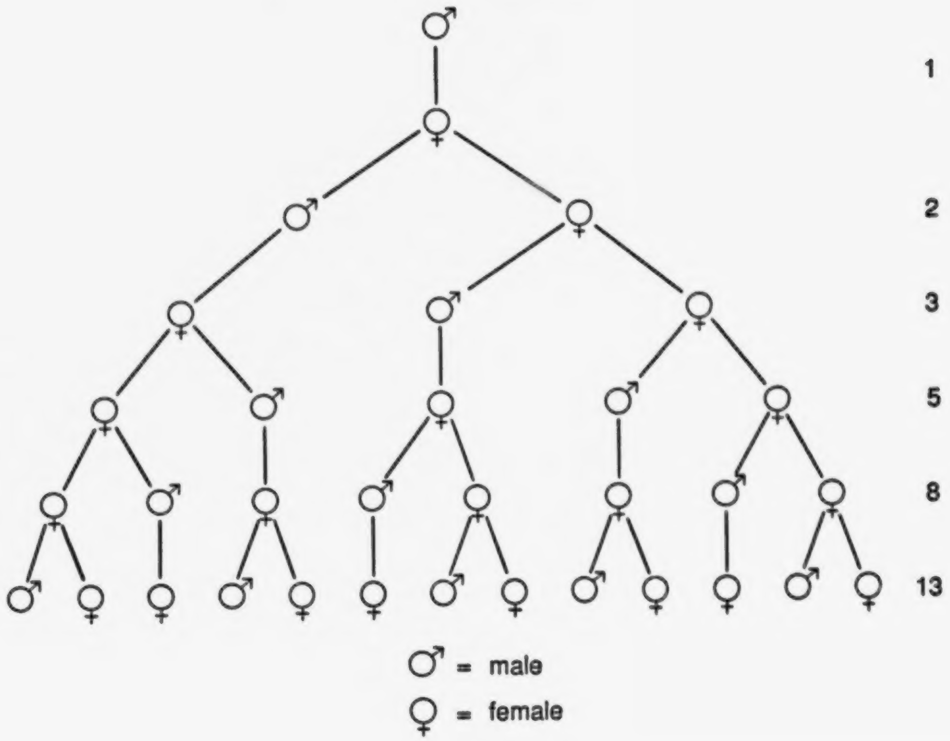


Figure B-7

PROBLEM 39 In Figure B-8, *ABCD* is a square with side 8. What is the area of the shaded part of the figure?

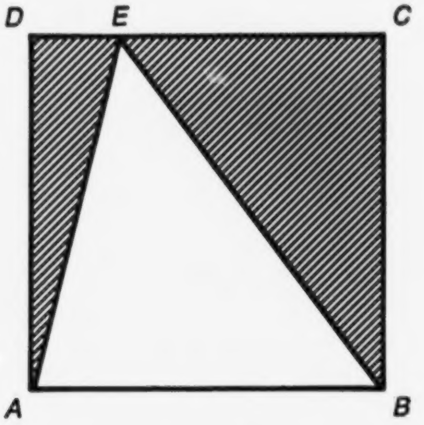


Figure B-8

Discussion The area of square *ABCD* is $AB \times BC$, or $8 \times 8 = 64$ square units. The area of the unshaded portion, triangle *ABE* is

$\frac{1}{2}(AB)(BC)$, since BC equals the altitude of the triangle. This gives 32 square units. Thus the shaded portion of the figure is also 32 square units. Some students might see that the sum of the areas of the shaded triangles is equal to the area of the unshaded triangle, since the sum of their bases $(DE + EC)$ equals the base of the triangle, AB, and they have the same altitudes.

PROBLEM 40 The function $f(n) = n^2$ generates the following sequence:

$$1, 4, 9, 16, 25, 36, 49, 64, \ldots$$

Find another function that produces the first seven terms of the sequence, but not the eighth.

Discussion Here is one possible answer:

$$f(n) = n^2 - (n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)$$

PROBLEM 41 A baker rolls out his dough in the morning and cuts it into 8 equal pieces, which he seasons. He then cuts each of these seasoned pieces into 4 equal parts. He bakes each of these into a loaf of bread that is $\frac{3}{4}$ of a foot long. If he were to place all of the loaves end-to-end, what would the total length be?

Discussion Simulate the action with a drawing. The baker made 8 × 4 or 32 loaves, with each loaf 9 inches long. Thus the length would be $32 \times 9'' = 288''$ or 24 feet.

PROBLEM 42 The drawing in Figure B-9 shows an arrangement of 7 squares. If the perimeter and the area of the figure are numerically equal, find a side of one square.

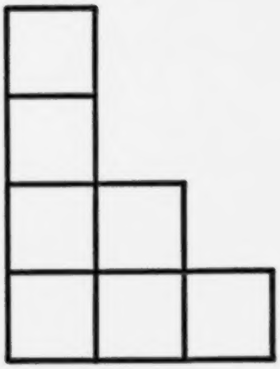


Figure B-9

Discussion Let a side of one square be x. Since the perimeter of the figure is numerically related to the area,

$$7x^2 = 14x$$
$$7x = 14$$
$$x = 2$$

A side of one square is 2 units.

PROBLEM 43 The factory planners have built stations for their night watchmen at points *A* and *B* in Figure B-10. They are now ready to install check-in boxes along the two intersecting walls, *RS* and *TU*. They wish to install these check-in boxes so that the watchmen walking from station *A* to the box on *RS* to the box on *TU* to station *B* will be making a minimum trip. What is the shortest path meeting these conditions?

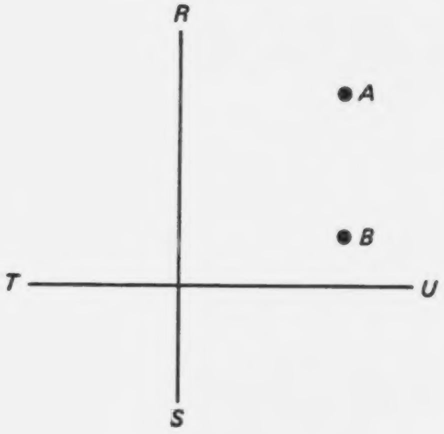


Figure B-10

Discussion This problem requires a double reflection. We first reflect *B* through *TU* to point *B′* (see Figure B-11). Then we reflect *B′* through *RS* to *B″*. Now we draw *AB″* to find the point for the check-in station along *RS* (point C). Then we draw *CB′* to find point *D* along *TU*.

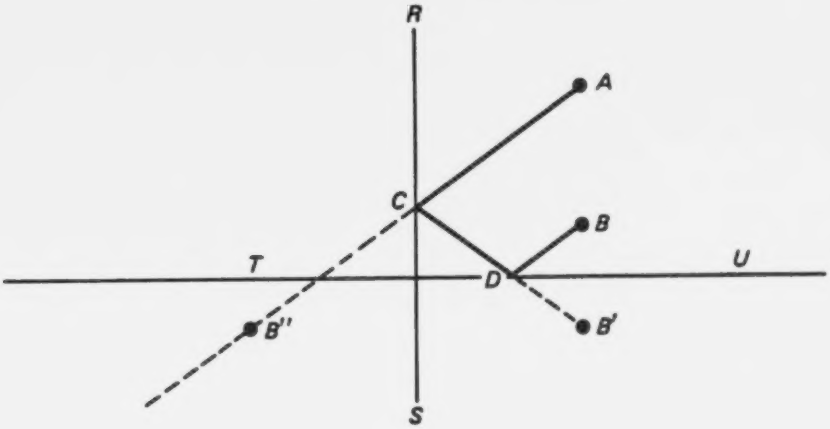


Figure B-11

PROBLEM 44 A dog food company decides that it wishes to change the size of the cylindrical can in which its product is sold. It decides that the new can should have the same volume as the old can but should be 1 inch taller than the old can. If the old can had a height of 4 inches and a diameter of 2.6 inches, what should be the diameter of the new can? (Find the answer to the nearest hundredth).

Discussion The volume of a cylinder is given by the formula $V = \pi r^2 h$. The new can will be 5 inches tall. If we let x represent the radius of the new can, its volume is equal to that of the old can. Thus,

$$\begin{aligned} \pi \cdot x^2 \cdot 5 &= \pi \cdot (1.3)^2 \cdot 4 \\ 5x^2 &= 6.76 \\ x^2 &= 1.352 \\ x &= 1.1627 \\ d = 2x &= 2.33 \qquad \text{(to the nearest hundredth)} \end{aligned}$$

PROBLEM 45 Tennis balls are packed in cylindrical cans of 3. The balls just touch the sides, top, and bottom of the can. How does the height of the can compare with the circumference of the top?

Discussion Represent the problem with a drawing, as in Figure B-12.

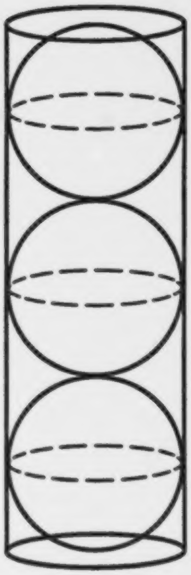


Figure B-12

The height of the can is $3d$.
The circumference of the top is πd.
The circumference is $3.14d$ and greater than the height, $3d$.

Another solution to the problem would consist of bringing in a can and actually having the students measure the required dimensions.

PROBLEM 46 Two girls, Jan and Tanya, have a job painting fence posts that line both sides of a path leading to a barn. Jan arrived early and had already painted 5 posts on one side when Tanya arrived. Before she started painting, Tanya said, "Jan, I'm left-handed and it's easier for me to paint this side of the path while you paint the other side." Jan agreed and went over to the other side of the path. Tanya painted all the posts on her side, then went across the path and painted 10 posts on Jan's side. This finished the job. If there were the same number of posts on each side of the path, who painted more posts and how many more?

Discussion Tanya painted 10 posts from Jan's side, plus her own row, minus the 5 posts Jan had already done. Thus Tanya painted an entire row plus 5 posts while Jan painted an entire row minus 5 posts. Tanya pained 10 more posts than did Jan.

PROBLEM 47 A merchant visited three fairs. At the first, he doubled his money and spent \$30. At the second, he tripled his money and spent \$54. At the third fair, he quadrupled his money and spent \$72. He then found that he had \$48. How much money did he start with?

Discussion An algebraic solution should be developed in three steps:

Step 1 $2x - 30$

Step 2 $3(2x - 30) - 54$

Step 3 $4[3(2x - 30) - 54] - 72 = 48$

Another solution would be to work backwards:

Step 1	*Step 2*	*Step 3*
48 + 72 = 120	30 + 54 = 84	28 + 30 = 58
120 ÷ 4 = 30	84 ÷ 3 = 28	58 ÷ 2 = 29

He began with \$29.

PROBLEM 48 There are four boys of different ages, heights, and weights. Al, the youngest, is shorter than Bob, the heaviest, who is younger than Carl, the tallest. If no boy occupies the same ranking in any two categories, how does Dan compare with the others?

Discussion Examining the clues leads to the following matrix:

	Age	*Height*	*Weight*
First	Dan	Carl	Bob
Second	Carl	Bob	Al
Third	Bob	Al	Dan
Fourth	Al	Dan	Carl

Dan is the oldest and the shortest and weighs second least.

PROBLEM 49 Find the units digit of 1492^{1989}

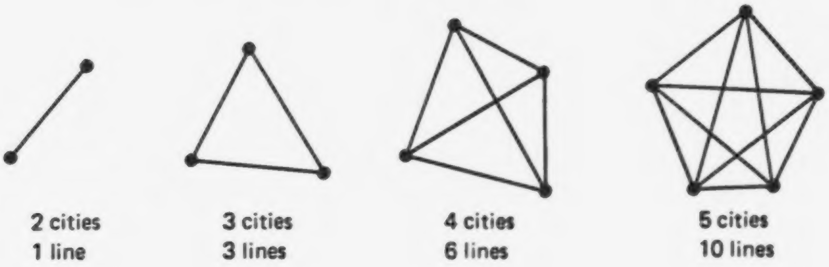
Discussion Successive powers of 2 reveal a mod 4 system for the final digits:

$$\begin{array}{lll} 2^1 = 2 & 2^5 = 32 & 2^9 = 512 \\ 2^2 = 4 & 2^6 = 64 & 2^{10} = 1024 \\ 2^3 = 8 & 2^7 = 128 & 2^{11} = 2028 \\ 2^4 = 16 & 2^8 = 256 & 2^{12} = 4096 \end{array}$$

Now we consider the exponent 1989. $1989 \equiv 1 \pmod 4$. Thus 1492^{1989} will end in a 2.

PROBLEM 50 There are 26 football teams in the National Football League. To conduct their annual draft, teams in each city must have a direct telephone line to each of the other cities. How many direct telephone lines must be installed by the telephone company to accomplish this?

Discussion Examine the problem by reducing the complexity. We can start with two cities, then three cities, four cities, etc., keeping track of the information and searching for a pattern. (See Figure B-13.)



Number of cities	1	2	3	4	5	. . .	n
Number of lines	0	1	3	6	10	. . .	

Figure B-13

Thus each city is connected to every other $(n - 1)$ city. Since every two cities share a line, our total will be

$$\frac{n(n-1)}{2}$$

For 26 teams, the total will be

$$\frac{(26)(25)}{2}$$

or 325 telephones.

PROBLEM 51 A paperboy agrees to deliver newspapers for one year. In return, he will receive a salary of $240 plus a new bike. He quits after 7 months, and receives $100 and the bike. What was the value of the bike?

Discussion Representing the salary algebraically, $240 + x$ would be the salary for one year, where x represents the value of the bicycle. Since he left after only 7 months,

$$\begin{aligned} \frac{7}{12}(240 + x) &= 100 + x \\ 7(240 + x) &= 1200 + 12x \\ 1680 + 7x &= 1200 + 12x \\ 480 &= 5x \\ 96 &= x \end{aligned}$$

The bike was worth $96.

Students may also reason that he should have received $20 per month plus $\frac{1}{12}$ of the bike per month. Thus, for 7 months, he received $140 + $\frac{7}{12}$ of the bike. Therefore, the missing $\frac{5}{12}$ of the bike was the extra $40, and $\frac{1}{12}$ of the bike would be $8. Thus the bike is worth $96.

PROBLEM 52 The checkerboard shown in Figure B-14 contains one checker. The checker can only move diagonally "up" the board along the white squares. In how many ways can this checker reach the square marked A?

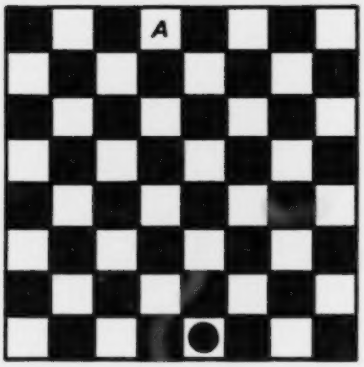


Figure B-14

Discussion

In Figure B-15, we have marked the number of ways the checker can reach the given square. For example, to reach the square labeled *B*, the checker might take exactly 3 different paths. Notice that these numbers form the Pascal Triangle, and thus we can compute that there are a total of 35 different paths the checker might take.

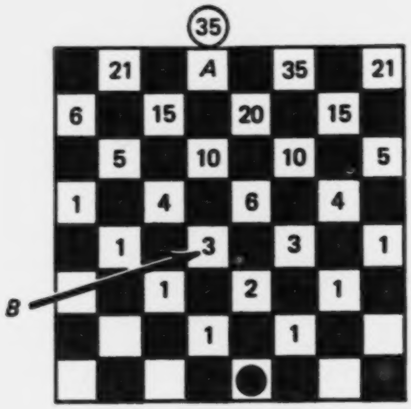


Figure B-15

PROBLEM 53

The numbers on the uniforms of the Granville Baseball Team all consist of two digits. Two friends on the team are also amateur mathematicians. They select their numbers so that the square of the sum of their numbers is the same as the four-digit number formed by their uniforms when they stand side by side. What are the numbers on their uniforms?

Discussion

Exhaustive listing with the use of a calculator provides a solution. The problem tells us that the square of the sum of the two numbers is a four-digit number. Thus, n^2 lies between 1,000 and 9,999. Use a calculator to identify all the perfect squares between the limits and see which satisfy the condition. There are only three possible answers:

$$\begin{array}{llll} 98\ \ 01 = (99)^2 = (98 + 01)^2 & & 98 & 01 \\ 30\ \ 25 = (55)^2 = (30 + 25)^2 & & 30 & 25 \\ 20\ \ 25 = (45)^2 = (20 + 25)^2 & & 20 & 25 \end{array}$$

Notice that we can eliminate the first possible answer (98 and 01) because 01 would not appear as a two-digit number on a uniform.

PROBLEM 54

At a major automobile race, Speedy McDriver averaged 110 miles per hour for the first half of the distance and 130 miles per hour for the second half. Meanwhile, Cannonball Casey drove the entire race at a constant speed of 120 miles per hour. Who won the race?

Discussion

Some students will average the given rates of 130 miles per hour and 110 miles per hour to arrive at an average rate of 120 miles per hour. This is wrong! The rates are *themselves* averages, and one does not average averages, unless the times are the same. Let's set up two expressions for the time it took each driver and compare them:

Speedy McDriver

R ×	T =	D
130	$D/130$	D
110	$D/110$	D

$$\frac{D}{130} + \frac{D}{110}$$

$$\frac{24D}{1430}$$

Cannonball Casey

R ×	T =	D
120	$D/120$	D
120	$D/120$	D

$$\frac{D}{120} + \frac{D}{120}$$

$$\frac{24D}{1440}$$

The time taken by Speedy McDriver was $24D/1430$. The time taken by Cannonball Casey was $24D/1440$. Thus, Cannonball Casey took less time than McDriver to finish the race. The winner was Cannonball Casey.

PROBLEM 55 Four married couples went to the baseball game last week. The wives' names are Carol, Sue, Jeanette, and Arlene. The husbands' names are Dan, Bob, Gary, and Frank. Bob and Jeanette are brother and sister. Jeanette and Frank were once engaged, but broke up when Jeanette met her husband. Arlene has a brother and a sister, but her husband is an only child. Carol is married to Gary. Who is married to whom?

Discussion Prepare a 4 × 4 matrix like the one shown. Use the clues to eliminate the incorrect choices.

	Carol	Sue	Jeanette	Arlene
Frank	X	X	X	Yes
Gary	Yes	X	X	X
Bob	X	Yes	X	X
Dan	X	X	Yes	X

PROBLEM 56 Two couples were sitting on a park bench, posing for a picture. If neither couple wishes to be separated, what is the number of different possible seating arrangements that can be used for the photograph?

Discussion Simulate the action with a drawing. Let the letters *A*,*B* represent the first couple and the letters *C*,*D* represent the second couple. Remember that *A*,*B* is different from *B*,*A*. There will be 8 arrangements.

PROBLEM 57 Mr. Johnson was approached by his club for a contribution and was told that 10 members had already donated $10 each. He then wrote out a check for a sum such that his contribution exceeded the average contribution of the 11 donors by $20. What was his contribution?

Discussion The total donation by the ten earlier contributors was 10 × 10 or 100 dollars. If we let x represent Mr. Johnson's contribution, then

$$\frac{100 + x}{11} = x - 20$$

$$\begin{aligned} 100 + x &= 11x - 220 \\ 320 &= 10x \\ 32 &= x \end{aligned}$$

Mr. Johnson donated \$32.

Alternatively, we might reason in the following manner. The 10 original contributors donated \$100, or \$10 each. Mr. Johnson raises their average donation by \$20/10, or \$2. Thus he must contribute his \$10, plus \$2 for his own increase, plus \$20 to make up their difference. \$10 + \$2 + \$20 = \$32.

PROBLEM 58 A man persuaded his friend to work on a job for 36 consecutive days, under the following salary scale: he was to receive \$8 for every day he worked, but he would lose \$10 for every day he did not work. At the end of the 36 days, neither owed the other anything. How many days did the friend work?

Discussion Let x = the number of days the friend worked.
Then $(36 - x)$ = the number of days he did not work.

$$\begin{aligned} 8x &= 10(36 - x) \\ 8x &= 360 - 10x \\ 18x &= 360 \\ x &= 20 \end{aligned}$$

He worked 20 days.

PROBLEM 59 Find the area of the shaded portion of Figure B-16, formed by the four congruent semicircles drawn in the rectangle as shown.

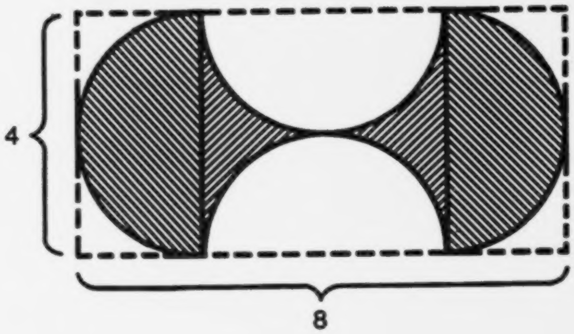


Figure B-16

Discussion Since the four semicircles are congruent, each must have a diameter of 4. The students should observe that the unshaded semicircles will be exactly filled in by the two shaded semicircles. Thus, the area is 4 × 4 = 16 square units.

PROBLEM 60 At a large picnic, there were 45 dishes served altogether. Every 3 people shared a dish of cole slaw between them. Every 4 people shared a dish of potato salad between them. Every 6 people shared a dish of hot dogs between them. How many people were at the picnic?

Discussion The number of people at the picnic must be a multiple of 3, 4, and 6. The lowest common multiple of these numbers is 12. Guess and test with multiples of 12 to find the answer. There were 60 people at the picnic, since

Every 3 people share a dish of cole slaw:	20 dishes
Every 4 people share a dish of potato salad:	15 dishes
Every 6 people share a dish of hot dogs:	10 dishes

20 + 15 + 10 gives a total of 45 dishes.

PROBLEM 61 A travel agency offers a charter trip to Yellowstone National Park. They charge $300 per person if they can fill all 150 places. If not, the price per ticket is increased by $5 for every place not sold. How many tickets should be sold to give the agency the maximum income for the trip?

Discussion The income depends on the number of tickets sold and the price per ticket. Let's make a table.

No. tickets not sold	*No. tickets sold*	*Price per ticket*	*Income*
10	140	$350	$49,000
20	130	400	52,000
30	120	450	54,000
40	110	500	55,000
45	105	525	55,125
50	100	550	55,000
60	90	600	54,000

Maximum income occurs when 45 tickets are *not* sold, or 105 tickets are sold.

For those students who have had some elementary calculus, the problem reduces to a traditional maximum/minimum problem. An equation representing income should be developed:

x = number of tickets not sold
I = total income
$I = (150 - x)(300 + 5x)$
$I = -5x^2 + 450x + 45000$

The first derivative yields $450 - 10x$. Setting this equal to zero yields $x = 45$.

PROBLEM 62 Solve for x:

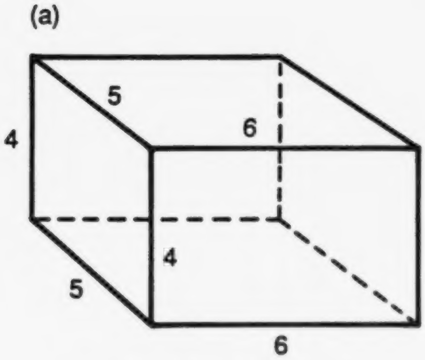
$$4^x + 4^{(x+1)} = 40$$

Discussion Factoring the left side yields:

$$\begin{aligned} 4^x(1 + 4^1) &= 40 \\ 4^x(5) &= 40 \\ 4^x &= 8 \\ (2^2)^x &= 2^3 \\ 2x &= 3 \\ x &= \frac{3}{2} \end{aligned}$$

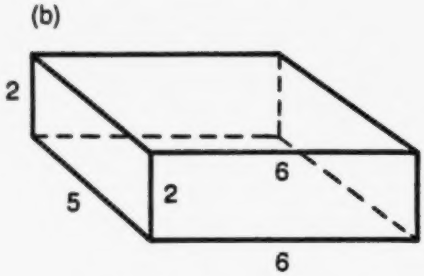
PROBLEM 63 A block of cheese 4 inches by 5 inches by 6 inches is dipped into a vat of wax and covered on all sides. The cheese is then cut into one-inch cubes. How many of these cubes will not have wax on them?

Discussion Simulate the "peeling" of the faces with a series of drawings as shown in Figure B-17.



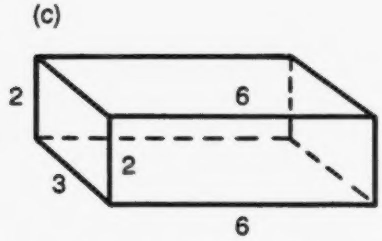
Remove top and bottom:

6 x 5 = 30
6 x 5 = 30
60



Remove front and back:

6 x 2 = 12
6 x 2 = 12
24



Remove left and right:

3 x 2 = 6
3 x 2 = 6
12

Figure B-17

The total number of cubes will be 4 × 5 × 6 or 120. The number of cubes with wax will be 96. Thus there will be 24 cubes with no wax on them. (This is easily verified, since the final block of cheese after the three "peelings" will be 4 × 2 × 3 and will produce 24 cubes with no wax on them.)

PROBLEM 64 Find the natural numbers less than 200 that have exactly three factors.

Discussion Since factors occur in pairs, the only numbers with 3 factors would be the perfect squares. The numbers would be 4, 9, 16, 25, . . . , 196.

PROBLEM 65 A merchant buys his goods at 25% off the list price. He then marks the goods so that he can give his customers a discount of 20% on the marked price but still make a profit of 25% on the selling price. What is the ratio of marked price to list price?

Discussion This problem is best done by algebra. Caution students that a discount of 25% and a profit of 25% need not be the same thing, since they are usually based on different amounts. Notice that they need not solve for the marked price—only the relationship.

$$\begin{aligned}
\text{List price} &= x \\
\text{Merchant's cost} &= .75x \\
\text{Marked price} &= MP \\
.8MP - .75x &= (.25x)(.8MP) \\
.8MP - .75x &= .2MP \\
\frac{MP}{x} &= \frac{.75}{.6} \\
&= \frac{75}{60} \\
&= \frac{5}{4}
\end{aligned}$$

PROBLEM 66 When Jane mailed a letter, postal rates for a first-class letter were 20¢ for the first ounce or fraction thereof, and 17¢ for each additional ounce or fraction. If Jane spent $1.75 to mail a first-class letter, what was the weight of the letter?

Discussion Here is an opportunity for students to see a bona fide step function. They should draw a graph representing the costs of mailing first-class letters. (See Figure B-18.)

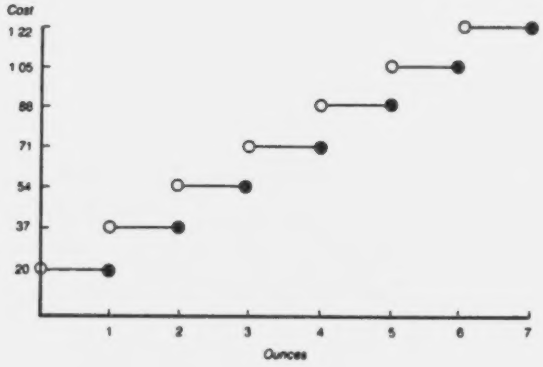


Figure B-18

PROBLEM 67 Steve is responsible for keeping the fish tank in the Seaside Aquarium Shop filled. One of the 50-gallon tanks has a small leak, and, along with evaporation, it loses 2 gallons of water each day. Every three days, Steve adds 5 gallons

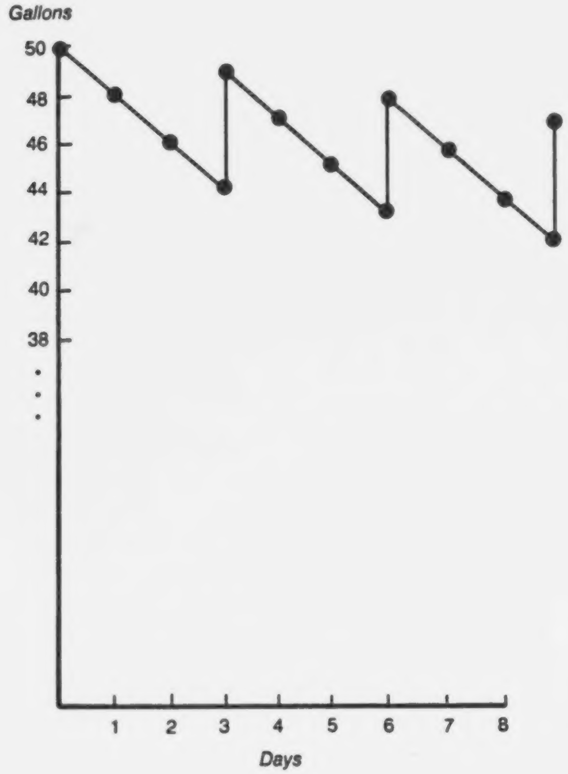


Figure B-19

of water to the tank. After 30 days, he fills it. How much water does Steve add on the thirtieth day to fill the tank?

Discussion There are several solutions to this interesting problem. The first is to notice that there is a net loss of 1 gallon every 3 days, which means a loss of 10 gallons during the 30-day interval. Thus, Steve adds 10 gallons on the thirtieth day.

A more interesting solution is achieved by drawing a graph of the situation, which turns out to be a sawtooth graph as shown in Figure B-19.

PROBLEM 68 Mr. Lopez, who is 6 feet tall, wants to install a mirror on his bedroom wall that will enable him to see a full view of himself. What is the minimum-length mirror that will serve his needs, and how should it be placed on his wall?

Discussion This problem is resolved by applying a well-known theorem of geometry. A drawing such as the one in Figure B-20 is the vital solution strategy.

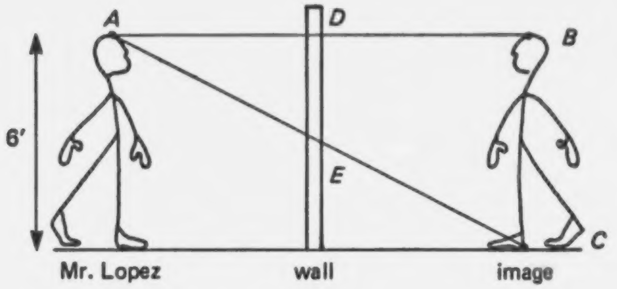


Figure B-20

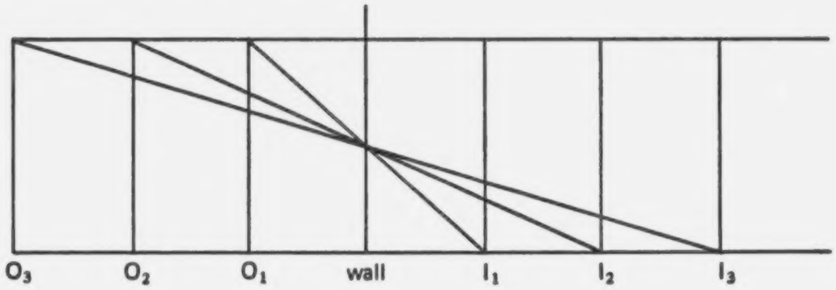


Figure B-21

Since the object, the image, and the wall are each perpendicular to the floor, the lines are parallel. Thus triangle *ABC* has segment *DE* parallel to *BC* and equal in length to half of *BC*. The conclusion is that the mirror must be 3 feet in length and hung so that its lower edge is 3 feet from the floor. Figure B-21 shows that the distance Mr. Lopez stands from the mirror will not affect the answer.

PROBLEM 69 In a game called crossball, a team can score either 3 points or 7 points. Which scores can a team *not* make?

Discussion We set up a table to organize our discussion:

Not possible	1	2		4	5			8			11			
Possible			3			6	7		9	10		12	13	14 . . .

Apparently a team can make every score after 11, since 3 can be added to each of the three successive scores 12, 13, and 14.

Notice that we can extend this problem as follows: Given any two relatively prime numbers, what is the greatest counting number that cannot be expressed using these numbers singly, repeatedly, or in combination? (This leads to the formula $ab - (a + b)$ as the method for determining the answer.)

PROBLEM 70 Find the smallest perfect square of three digits such that the sum of these digits is not a perfect square.

Discussion An organized list would be helpful. Since the perfect square is to contain three digits, the number itself must lie between 10 and 31 inclusive.

$10^2 = 100$ the sum of the digits is 1

$11^2 = 121$ the sum of the digits is 4

$12^2 = 144$ the sum of the digits is 9

Continuing this list reveals that the square of 16 (256) is the required number, since $2 + 5 + 6 = 13$, not a perfect square.

PROBLEM 71 Find the value of

$$\frac{\sin 10^\circ \ \cos 10^\circ \ \tan 10^\circ \ \csc 10^\circ \ \sec 10^\circ \ \cot 10^\circ}{\sin 20^\circ \ \cos 20^\circ \ \tan 20^\circ \ \csc 20^\circ \ \sec 20^\circ \ \cot 20^\circ}$$

Discussion Some students will go directly to the tables for the values of these functions. Others will use a calculator. A more elegant method of solution is to realize that $\sin x^\circ \csc x^\circ = 1$, $\cos x^\circ \sec x^\circ = 1$, and $\tan x^\circ \cot x^\circ = 1$. Thus the fraction reduces to $\frac{1}{1}$, or simply 1.

PROBLEM 72 Three cylindrical oil drums of 2-foot diameter are to be securely fastened in the form of a "triangle" by a steel band. What length of band will be required?

Discussion The careful drawing of the situation shown in Figure B-22 is the fundamental strategy used here. Notice that the band consists of three straight sections (tangents) and three curved sections (arcs). The straight sections form rectangles with the line of centers, and thus each equals $2r$ or 2 feet. The curved sections are each arcs whose central

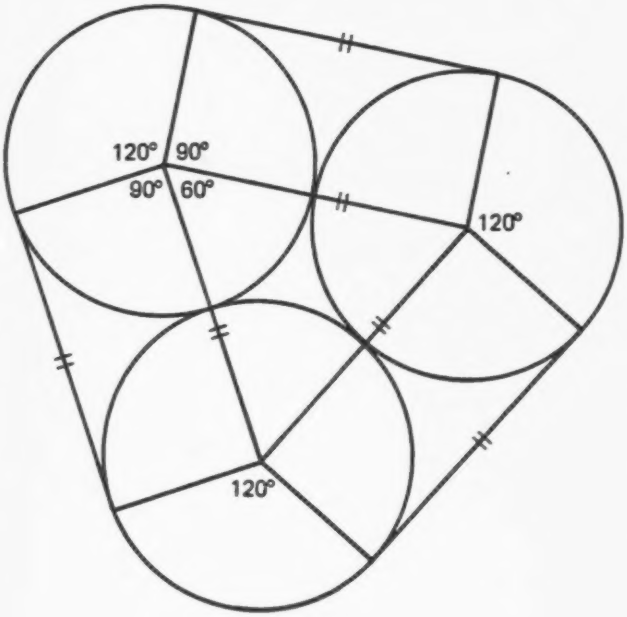


Figure B-22

angles are 120°; thus, each arc is $\frac{1}{3}$ of a circle, or $(\frac{1}{3})(\pi d)$. The total length is

$$3(2) + 3(\tfrac{1}{3}\pi d) = 6 + \pi d$$

Since $d = 2$ feet, the length of the steel band will be $6 + 2\pi$ feet, or approximately 12.28 feet.

Notice that this problem can be generalized for n barrels and also extended to barrels of different diameters.

PROBLEM 73 Find four numbers a, b, c, and d such that $a^2 + b^2 + c^2 = d^2$. (These are called Pythagorean quadruples.)

Discussion One solution is to build a series of right triangles with integral sides, as shown in Figure B-23.
The figure reveals two possible answers:

$$3^2 + 4^2 + 12^2 = 13^2$$
$$5^2 + 12^2 + 84^2 = 85^2$$

Many possible answers exist.

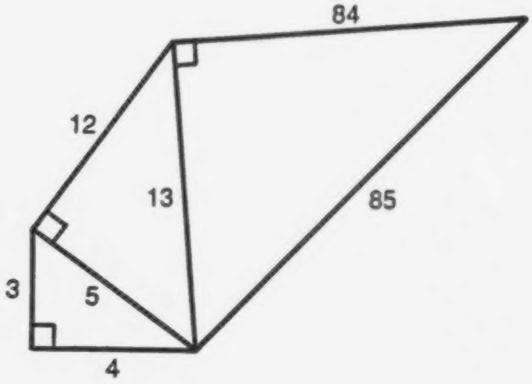


Figure B-23

PROBLEM 74 A meter records voltage readings between 0 and 20 volts. If the average value for the three readings on the meter was 16 volts, what could have been the smallest possible reading?

Discussion If the average of the three readings was 16, then the total of the three readings must have been $3 \times 16 = 48$. Since the reading can only be a maximum of 20, we use $2 \times 20 = 40$, leaving a minimum possible reading of 8 volts.

PROBLEM 75 An oil storage tank recorded $\frac{1}{12}$ full when its supply was replenished. After filling it, the tank recorded $\frac{7}{8}$ full. If 1,197 gallons were actually delivered, how many gallons of oil were in the tank before the delivery?

Discussion Since the tank was $\frac{7}{8}$ full at the end and $\frac{1}{12}$ full at the beginning, their difference, $\frac{19}{24}$, represents the part of the tank that was filled with 1,197 gallons. Dividing by 19, we find that there are 63 gallons per 24th, or 126 gallons in $\frac{1}{12}$th of a tank. An alternate method would be first to determine the capacity of the full tank. This is done by multiplying 1,197 by $\frac{24}{19}$.

PROBLEM 76 Students in Mr. Josephson's Physical Education class are spaced evenly around a circle and then told to count off beginning with 1. Student number 16 is directly opposite student number 47. How many are in Mr. Josephson's class?

Discussion Geometrically, the students would represent points on a circle which are the opposite ends of a diameter. Thus, the number of students in the class must be an even number. If 16 and 47 are the ends of a diameter, there will be 30 people between 16 and 47 and 30 people between 47 and 16. There will be $30 + 30 + 2$ or 62 people on the circle. A challenge for the students would be to generalize the situation with a formula.

PROBLEM 77 The Shah used to give gold pieces to his sons and daughters every year on the occasion of his own birthday. He would give each son as many gold pieces as he had sons, and he would give each daughter as many gold pieces as he had daughters. One year he gave them a total of 841 gold pieces. How many sons and how many daughters did he have?

Discussion If x represents the number of sons and y represents the number of daughters, then we obtain the equation

$$x^2 + y^2 = 841$$

Since the answers must be integral, there are four possibilities:

Section B

Sons	0	29	20	21
Daughters	29	0	21	20

However, since he had both sons and daughters according to the problem, only two possibilities will satisfy.

PROBLEM 78 What is the angle between the hands of a clock at 2:15?

Discussion Look at the drawing of a clock shown in Figure B-24. The hour hand is $\frac{3}{4}$ of the distance between the 2 and the 3. Since there are 30 degrees between the 2 and the 3, the hour hand has moved through $22\frac{1}{2}$ degrees. Thus there are $67\frac{1}{2}$ degrees between the hands, since the minute hand is exactly on the 3.

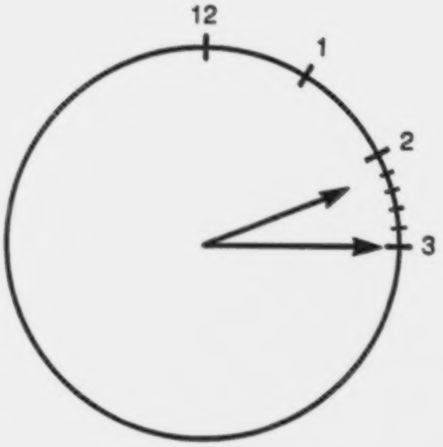


Figure B-24

PROBLEM 79 At what times between 7:00 and 8:00 will the hands of a clock form a right angle?

Discussion The angle formed by the hands of a clock will repeat 11 times in a 12-hour span, or every $1{:}05\frac{5}{11}$ hours. There will be a perfect right angle at 3:00, and again at

$$4{:}05\tfrac{5}{11},\ 5{:}10\tfrac{10}{11},\ 6{:}16\tfrac{4}{11},\ \text{and}\ 7{:}21\tfrac{9}{11}$$

But there is also a perfect right angle at 9:00. There will be another one at $9{:}00 - 1{:}05\frac{5}{11}$ or $7{:}54\frac{6}{11}$.

PROBLEM 80 A high school geometry class was given the problem of finding the height of the flagpole in the school yard with only a mirror and a measuring tape. Several of the students successfully answered the question. How did they solve the problem?

Discussion Draw a diagram of the situation, as in Figure B-25.

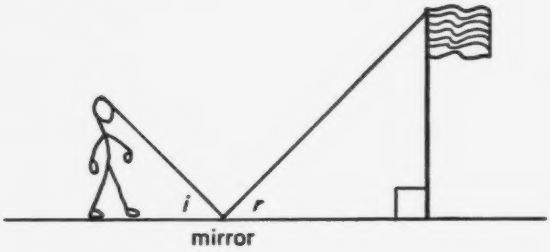


Figure B-25

Since the two triangles are similar (the angle of incidence, *i*, equals the angle of reflection, *r*), a simple proportion reveals the answer.

PROBLEM 81 During the week, a furniture maker made 103 matching legs, some to be used for four-legged chairs and the others to be used for three-legged stools. How many possible combinations of chairs and stools can she make?

Discussion We are looking for multiples of 3 that, when subtracted from 103, leave multiples of 4. Prepare a table:

Stools	*Legs*	*Chairs*	*Legs*
33	99	1	4
29	87	4	16
25	75	7	28
21	63	10	40
17	51	13	52
13	39	16	64
9	27	19	76
5	15	22	88
1	3	25	100

There are 9 combinations possible.

An alternate solution is to graph $3x + 4y = 103$ and see which ordered pairs work. Notice that this problem lends itself to the writing of a computer program.

PROBLEM 82

A new car is being tested on a special circular track that is exactly 1 mile in length. The test driver goes through the first lap at exactly 30 miles per hour. How fast must she drive during the second lap in order to average 60 miles per hour for the entire test run?

Discussion

At 30 miles per hour, it takes 2 minutes to go 1 mile. Thus it took 2 minutes to cover the first lap. At an average speed of 60 miles per hour, it would take 2 minutes to cover 2 laps. Since she had already used up 2 minutes on the first lap, she cannot meet the given conditions.

PROBLEM 83

A woman went into a bank to cash a paycheck. As he handed over the money, the bank teller, by mistake, gave her dollars for cents and cents for dollars. She put the money away without counting it, and spent a nickel on the way home. She then found out that she had exactly twice as much money as the original check called for. She had no money in her pocket when she went into the bank. What was the exact amount of the check?

Discussion

Let c = the number of cents
d = the number of dollars.
Then the original check was $100d + c$ (in cents), and the amount she mistakenly received was $100c + d$ (in cents).

$$\begin{aligned} 100c + d - 5 &= 2(100d + c) \\ 100c + d - 5 &= 200d + 2c \\ 98c - 199d &= 5 \end{aligned}$$

Since both c and d must be integral, we arrive at $63 = c$ and $31 = d$. The original check was for $31.63.

PROBLEM 84

Whenever a cattle drive crosses the Bar X ranch, the owner charges 10¢ each for riderless animals and 25¢ for a cowboy and horse combination. Last week he counted a total of 4,248 legs and 1,078 heads. How much money did the owner of the Bar X ranch collect?

Discussion

Let x = the number of riderless animals
y = the number of cowboy/horse units.

Then we obtain the set of equations

$$x + y = 1078$$
$$4x + 6y = 4248$$

which yield $x = 1014$ and $y = 64$.
There are 1,014 riderless animals at 10¢ each = \$101.40, and there are 64 cowboy/horse combinations at 25¢ each = \$16.00. The total cost was \$117.40.
The problem can also be solved by using "guess and test" with the appropriate table.

PROBLEM 85 Michelle was driving along a highway when she noticed that her odometer read 15,951 miles, a palindrome. She was quite surprised to see that the odometer showed another palindrome in exactly two hours. How fast was Michelle driving?

Discussion The first digit (and the last one), 1, could not have changed in just 2 hours. However, the second (and the fourth) changed to a 6. Thus the odometer read 1 6 _ 6 1 after two hours. If we substitute 0, the car drove 110 miles or 55 miles per hour. If we substitute 1, the car drove 210 miles in 2 hours or 105 miles per hour. Similarly, 2, 3, 4, 5, . . . , all yield answers that are much too fast. Thus she was driving at 55 miles per hour.

PROBLEM 86 Susan jumped into her canoe and paddled upstream for one mile. She continued for another 15 minutes, then turned around and paddled downstream, arriving at her starting point in exactly one hour. How fast is the current of the stream?

Discussion Let x = her rate in still water
y = the rate of the stream.

R ×	T =	D
$x - y$	?	1
$x - y$	$\frac{1}{4}$	$\frac{x - y}{4}$
$x + y$	1	$x + y$

$$x + y = 1 + \frac{x - y}{4}$$

$$4x + 4y = 4 + x - y$$

$$3x + 5y = 4$$

If we let $y = 0$, then $x = \frac{4}{3}$.
If we let $y = \frac{1}{2}$, then $x = \frac{1}{2}$. (But this would be impossible, since Susan would be standing still when she moved against the current). Thus,

$$0 < y < \frac{1}{2}$$

$$\frac{4}{3} > x > \frac{1}{2}$$

The rate of the stream is between 0 and $\frac{1}{2}$ mile per hour.

PROBLEM 87 During the recent census, a man told the census taker that he had three children. When asked their ages, he replied, "The product of their ages is 72. The sum of their ages is the same as my house number." The census taker ran to the door and looked at the house number. "I still can't tell," she complained. The man replied, "Oh, that's right. I forgot to tell you that the oldest one likes chocolate pudding." The census taker promptly wrote down the ages of the three children. How old are they?

Discussion List all the combinations of three numbers whose product is 72, together with their sums:

1-1-72 = 74
1-2-36 = 39
1-3-24 = 28
1-4-18 = 23
1-6-12 = 19
1-8- 9 = 18
2-2-18 = 22
2-3-12 = 17
2-4- 9 = 15
2-6- 6 = 14
3-3-8 = 14
3-4-6 = 13

Since there was still a question after seeing the sum of the ages (the house number), there had to be more than one set of factors whose sum equaled this number (3-3-8 and 2-6-6). Since 2-6-6 does not yield an "oldest child" but 3-3-8 does, the ages of the children must have been 3, 3, and 8.

PROBLEM 88 A 6-foot-tall man looks at the top of a flagpole making an angle of 40° with the horizontal. The man stands 50 feet from the base of the flagpole. How high is the flagpole, to the nearest foot?

Discussion A drawing (Figure B-26) reveals that the problem can easily be solved by use of trigonometry. However, students must realize that the horizontal line of sight is 6 feet above the ground.

Thus, we set up the following equation:

$$\tan 40^\circ = \frac{x}{50}$$

$$.8391 = \frac{x}{50}$$

$$41.95 = x$$

The flagpole is 47.95 or 48 feet high to the nearest foot.

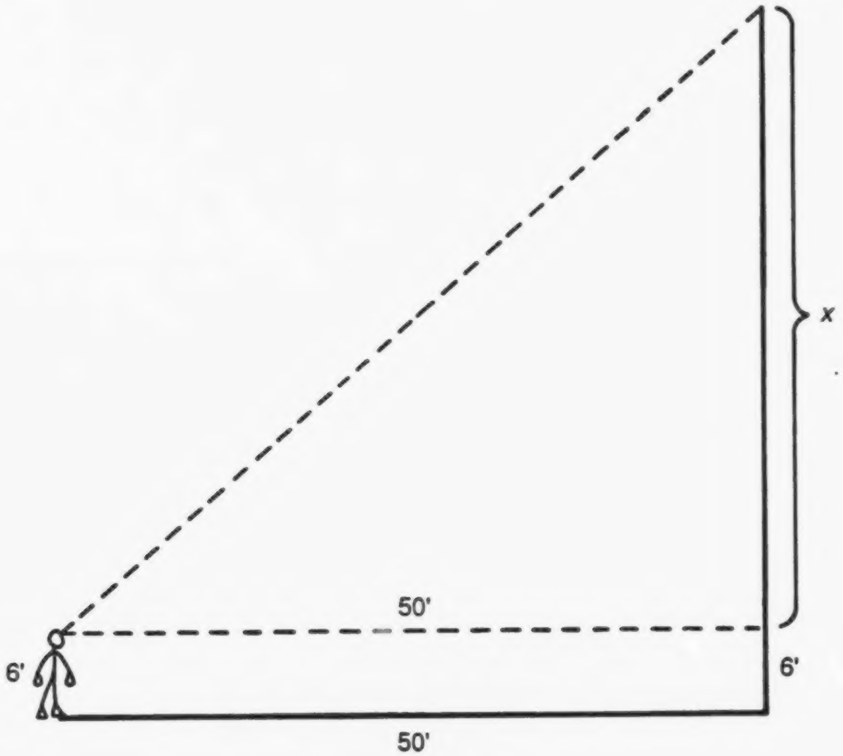


Figure B-26

PROBLEM 89 The shuttle service has a train going from New York to Boston and from Boston to New York, every hour on the hour. The trip from one city to another takes $4\frac{1}{2}$ hours. How many of the Boston-to-New York trains will pass you on your trip from New York to Boston if you leave New York at 9:00?

Discussion Simulate the action with a table. Remember that when you leave New York for Boston, there will be several trains already underway on a trip to New York from Boston.

Leaves Boston	*Arrives New York*
5:00	9:30
6:00	10:30
7:00	11:30
8:00	12:30
9:00	1:30
10:00	2:30
11:00	3:30
12:00	4:30
1:00	5:30

A total of 9 trains will pass you on the trip.

PROBLEM 90 Figure B-27 shows circle O, with diameter $= 2a$ and chord $\overline{YZ} = a$. Find $m < YXZ$

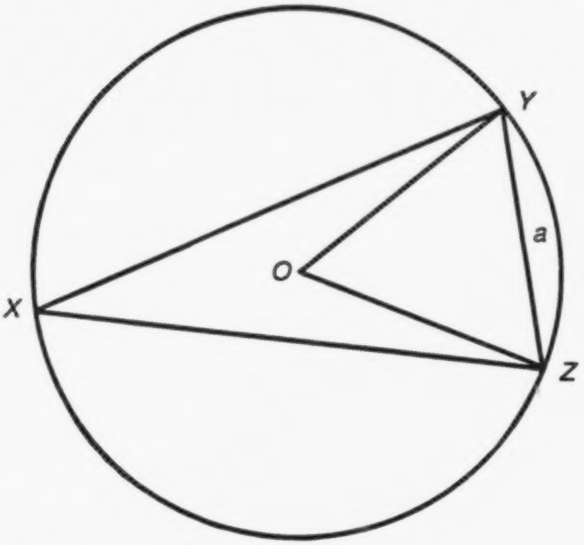


Figure B-27

Discussion Since diameter = $2z$, radii $\overline{OY}$ and $\overline{OZ}$ each = a. This makes triangle OYZ equilateral, and the measure of angle YOZ is 60°, as is the measure of YZ. Thus $m < YXZ$ is $\frac{1}{2}(YZ)$ or 30 degrees.

PROBLEM 91 Sammy Bigfoot jumped onto one end of a 13-foot tree trunk that was lying on the top of a hill. The impact of Sammy's jump caused the log to begin rolling downhill. As it rolled, Sammy was able to keep himself upright and slowly walked across to the other end of the log. He reached the other end just as the log came to rest at the bottom of the hill, exactly 84 feet from where it began to roll. If the diameter of the log is exactly 2 feet, how far did Sammy Bigfoot walk?

Discussion A diagram shows that Sammy's path is actually the diagonal of a rectangle with sides 13 feet and 84 feet. (See Figure B-28.) If we set up the relationship

$$x^2 = 13^2 + 84^2$$
$$x^2 = 169 + 7056$$
$$x^2 = 7225$$
$$x = 85$$

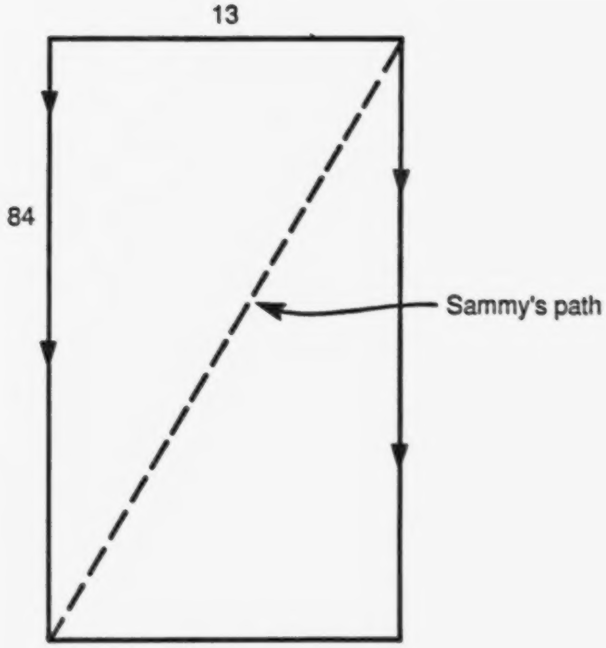


Figure B-28

Sammy walked 85 feet. Note that the diameter of the tree trunk has nothing to do with the problem. It is merely a distractor.

PROBLEM 92 A man walks for a total of 5 hours. First he walks along a level road, then he walks up a hill. At the top of the hill, he turns around and walks back to his starting point along the same route. Uphill, he walks at 3 kilometers per hour; downhill he walks at 6 kilometers per hour. How far did the man walk?

Discussion If we "unfold" his return trip along CB' and $B'A'$ (see Figure B-29), we have 4 distances to deal with.

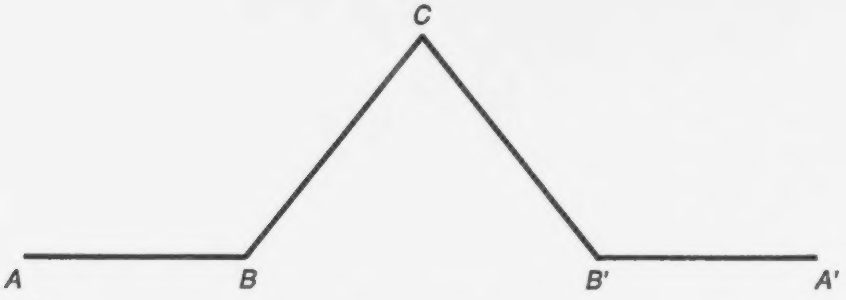


Figure B-29

Let $AB = x$, $BC = y$, $CB' = y$ and $B'A' = x$. Now set up a time equation:

$$\frac{x}{4} + \frac{x}{4} + \frac{y}{6} + \frac{y}{3} = 5$$
$$3x + 3x + 2y + 4y = 60$$
$$6x + 6y = 60$$
$$x + y = 10$$

The man walked a total of 20 miles. Note that the distance is 20 miles regardless of the individual values of x and y.

PROBLEM 93 A piece of "string art" is made by connecting nails that are evenly spaced on the vertical axis to nails that are evenly spaced on the horizontal axis, using colored strings. The same number of nails must be on each axis. Connect the nail farthest from the origin on one axis to the nail closest to the origin on the other axis. Continue in this manner until all the nails are connected. How many intersections are there if you use 8 nails on each axis?

Discussion Students might draw the finished situation as shown in Figure B-30, and try to count the number of intersections. This procedure is correct. However, some of the intersections might not be readily visible, and thus difficult to count. A more artistic solution involves reducing the complexity of the situation to two nails on each axis, then 3 nails, 4 nails, 5 nails, etc., as in Figure B-31. A table is used to record the results:

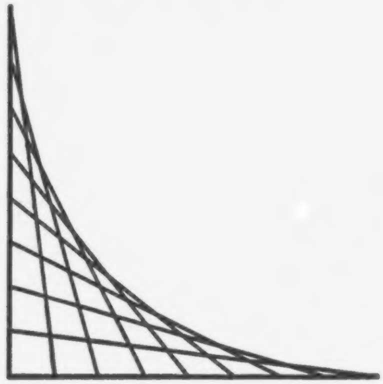
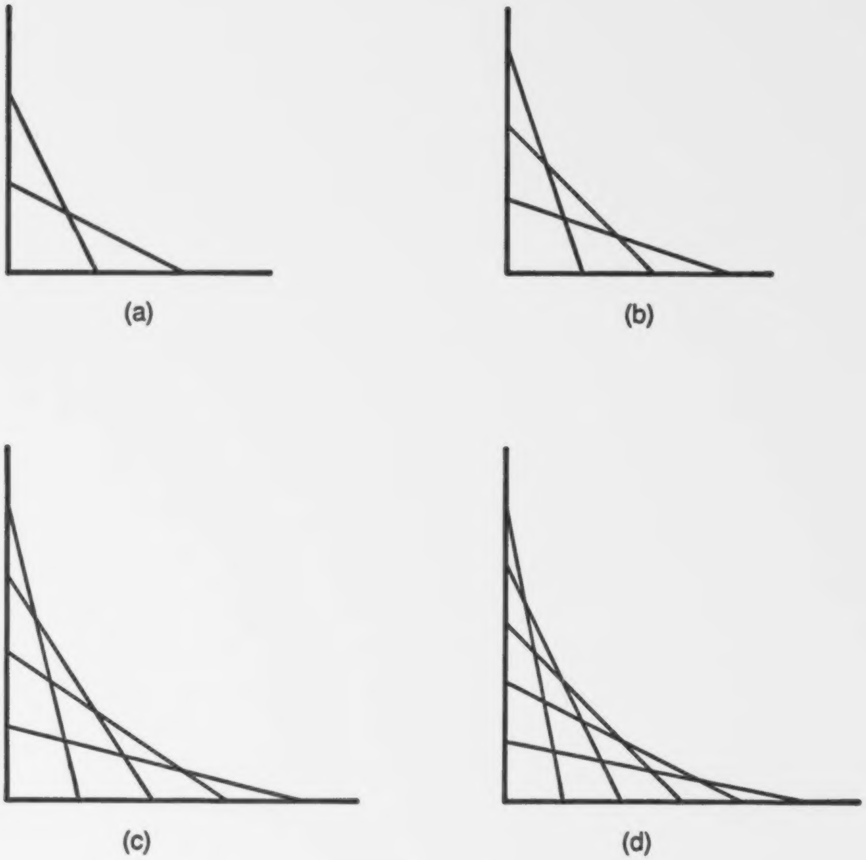


Figure B-30

Number of nails on each axis	*Number of intersections*	*(Differences)*
1	0	
2	1	1
3	3	2
4	6	3
5	10	4

Now the table can be continued until the entry for 8 nails on each axis (8 nails = 28 intersections). A further challenge for the students might be to develop an algebraic expression to find the number of intersections for any



(a) (b)

(c) (d)

Figure B-31

number of nails, n:

$$I = \frac{n(n-1)}{2}$$

PROBLEM 94 A cake in the form of a cube falls into a large vat of frosting and comes out frosted on all 6 faces. The cake is then cut into smaller cubes, each 1 inch on an edge. The cake is cut so that the number of pieces having frosting on 3 faces will be $\frac{1}{8}$ the number of pieces having no frosting at all. We wish to have exactly enough pieces of cake for everyone. How many people will receive pieces of cake with frosting on exactly 3 faces? On exactly 2 faces? On exactly 1 face? On no faces? How large was the original cake?

Discussion Make a table for a cube cake that was originally a 2 × 2 × 2 cake, a 3 × 3 × 3 cake, etc. Look for a pattern.

Cube	*3 frosted faces*	*2 frosted faces*	*1 frosted face*	*0 frosted faces*	
2 × 2 × 2	8	0	0	0	= 8
3 × 3 × 3	8	12	6	1	= 27
4 × 4 × 4	8	24	24	8	= 64
5 × 5 × 5	8	36	54	27	= 125
6 × 6 × 6	8	48	96	64	= 216

Notice that the number of small cubes with exactly 3 frosted faces is *always 8*. Thus the cube we are looking for is a cube having 8 times as many faces with no frosting or 64. This is the 6 × 6 × 6 cube as shown in our table. Thus there are 48 cubes with frosting on exactly 2 faces, 96 cubes with frosting on exactly 1 face, and a total of 216 small cubes in all. The original cake must have been a 6 × 6 × 6 cube.

PROBLEM 95 Irwin insists on adding two fractions by adding their numerators and then adding their denominators. Thus when he adds $\frac{9}{12}$ and $-\frac{1}{4}$, Irwin gets

$$\frac{9 + (-1)}{12 + 4} = \frac{8}{16} = \frac{1}{2}$$

To show that his method works, Irwin adds $\frac{9}{12}$ and $-\frac{1}{4}$ in the usual manner and gets

$$\frac{9 + (-3)}{12} = \frac{6}{12} = \frac{1}{2}$$

Are there other values for which this method of addition will produce the correct answer?

Discussion Let the fractions be represented by *a*/*b* and *c*/*d*. Since

$$\frac{a}{b} + \frac{c}{d} = \frac{a + c}{b + d}$$

merely adds symbols with no regard for meaning, we wish to find values for which

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} = \frac{a + c}{b + d}$$

Performing the algebra, we obtain $(ad + bc)(b + d) = bd(a + c)$, and finally,

$$c = \frac{ad^2}{b^2}$$

where b and $d \neq 0$. Integral values of a, b, c, and d that satisfy the above relationship will yield fractions to which Irwin's method may be applied.

PROBLEM 96 A commuter is in the habit of arriving at his suburban station each evening at exactly 6:00 P.M. His wife is always waiting for him with the car; she too arrives at exactly 6:00 P.M. She never varies her route or her rate of speed. One day he takes an earlier train, arriving at the station at 5:00 P.M. He decides not to call his wife, but begins to walk toward home along the route she always takes. They meet somewhere along the route, he gets into the car, and they drive home. They arrive home 10 minutes earlier than usual. How long had the husband been walking when he was picked up by his wife?

Discussion If we pursue the problem from the point of view of the husband's time, we cannot arrive at an answer easily. So consider the problem from the point of view of the wife—i.e., how long she was gone from the house. Since they arrive home a total of 10 minutes earlier than usual, the car was gone a total of 5 minutes less in each direction. Thus the husband had been walking a total of 55 minutes when he was picked up.

PROBLEM 97 Two girls wish to find the speed of a moving freight train as it passes by their town. They find that 42 railroad cars pass by the corner in 1 minute. The average length of a railroad car is 60 feet. At what speed is the train moving in miles per hour?

Discussion If 42 railroad cars pass by the corner in 1 minute, then 60 × 42 or 2,520 cars pass by in one hour. This is a total length of 2,520 × 60 or 151,200 feet per hour. Dividing this by 5,280, we obtain 28.6363. Thus, the speed of the train is approximately 29 miles per hour. (Again, use of the calculator enables the problem-solving aspect to be emphasized rather than the computational.)

PROBLEM 98

The probability of rolling a 2 on a standard pair of dice is $\frac{1}{36}$; the probability of rolling a 3 is $\frac{2}{36}$ (a 1–2 or a 2–1); and so on. How could you re-mark a pair of dice so that the probability of throwing each number from 1 through 12 was the same?

Discussion

If the probability of throwing each number from 1 through 12 is to be the same, then each number must have a probability of $\frac{3}{36}$. To allow you to roll a 1, the pair of dice would have to be marked with a 1 and a 0. To obtain the required probability of $\frac{3}{36}$, there must be 3 such situations. Thus the dice should be marked

Die 1	*Die 2*
0, 1, 2	1, 1, 1
3, 4, 5	7, 7, 7

PROBLEM 99

The sum of two numbers is 2, and the product of these same numbers is 3. Find the sum of their reciprocals.

Discussion

The usual algebraic solution to this problem is as follows. Let x and y represent the two numbers. Then

$$\begin{aligned} x + y &= 2 \\ xy &= 3 \end{aligned}$$

This yields a solution set

$$\left\{ \begin{array}{l} \frac{x}{x} = 1 + i\sqrt{2} \\ \frac{y}{y} = 1 - i\sqrt{2} \end{array} \right\}$$

Now, we must sum the reciprocals:

$$\frac{1}{1 + i\sqrt{2}} + \frac{1}{1 - i\sqrt{2}} = \frac{(1 - i\sqrt{2}) + (1 + i\sqrt{2})}{(1 + i\sqrt{2})(1 - i\sqrt{2})} = \frac{2}{3}$$

which is the correct answer. However, there is a much more artistic solution. The sum of the reciprocals of x and y is given by the expression

$$\frac{1}{x} + \frac{1}{y} = \frac{x + y}{xy}$$

But $x + y = 2$ and $xy = 3$. Thus we arrive at the answer $\frac{2}{3}$ with minimal calculation.

PROBLEM 100 Al, Bill, and Chris plan a picnic. Each boy spends exactly $9, and each buys sandwiches, ice cream, and soda. For each of these items, the boys spend a total of $9.00 jointly, although each boy splits his money differently and no boy spends the same amount of money for two different items. The greatest single expense was what Al paid for ice cream. Bill paid twice as much for sandwiches as for ice cream. How much did Chris spend for soda? (Consider only whole numbers of dollars.)

Discussion Use a 3 × 3 matrix as shown. The clue that Bill spends twice as much on sandwiches as on ice cream gives us two options: 4-2-3 or 2-1-6. But the largest single amount was spent by Al, which eliminates 2-1-6. Continuing in this manner reveals the answer: Chris spent $5 on soda.

	Sandwiches	Ice cream	Soda	
Al	2	6	1	= 9
Bill	4	2	3	= 9
Chris	3	1	5	= 9
	9	9	9	

PROBLEM 101 A bridge that spans a bay is 1 mile long and is suspended from two supports, one at each end. As a result, when it expands a total of 2 feet from the summer heat, the bridge "buckles" in the center, causing a bulge. How high is the bulge?

Discussion Draw a diagram of the bridge, showing the situation both before and after the expansion causes it to buckle. (See Figure B-32.)

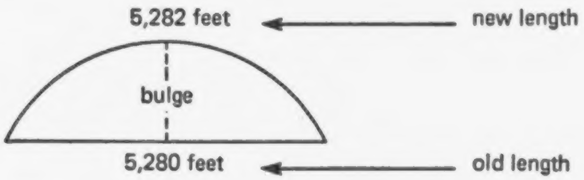


Figure B-32

We can now approximate the situation, using a diagram with two right triangles as in Figure B-33.

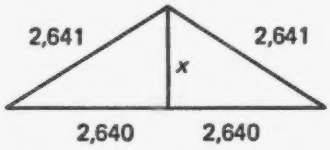


Figure B-33

The Pythagorean Theorem can now be applied to the right triangle:

$$x^2 + (2{,}640)^2 = (2{,}641)^2$$
$$x^2 + 6{,}969{,}600 = 6{,}974{,}881$$
$$x^2 = 5{,}281$$
$$x = 72.67$$

Thus, the bulge is approximately 73 feet high.

PROBLEM 102 A workman is fixing a broken cuckoo clock. The cuckoo comes out and stays out for a fixed number of minutes, then goes in and stays in for the same fixed number of minutes. The workman notices that the cuckoo came out at exactly 12:00. He looked up and noticed that it was *in* at 12:09; it was *out* when he looked up again at 12:17. It was *out* when he looked up at 12:58. What is the fixed interval of minutes? (Consider only integral answers.)

Discussion We can make a table that illustrates the "in-out" positions of the cuckoo. Since it came *out* at 12:00 and was *in* at 12:09, we reason that the unit of time must be 9 minutes or less. Several tables are shown:

	4-minute intervals	*5-minute intervals*	*7-minute intervals*	*9-minute intervals*
Out	12:00–12:04	12:00–12:05	12:00–12:07	12:00–12:09
In	12:04–12:08	12:05–12:10	12:07–12:14	12:09–12:18
Out	12:08–12:12	(No)	12:14–12:21	(No)
In	(No)		12:21–12:28	
Out			12:28–12:35	
In			12:35–12:42	
Out			12:42–12:49	
In			12:49–12:56	
Out			12:56– 1:03	

The only table that is consistent with the given data reveals that the time interval is 7 minutes.

PROBLEM 103 Brad and Dana had the same grade on their last exams in geometry; it was the highest grade for each of them, as well. The grade raised Brad's average from an 83 to an 86. It also raised Dana's average from an 88 to a 90. How many exams have they had prior to this last one?

Discussion An algebraic approach to the problem yields a pair of equations that can be solved simultaneously:

y = their score on the last exam
n = the number of exams prior to this one

Then,

$$\begin{array}{c|c}
\textit{Brad} & \textit{Dana} \\
\dfrac{83 \cdot n + y}{n + 1} = 86 & \dfrac{88 \cdot n + y}{n + 1} = 90 \\
83n + y = 86n + 86 & 88n + y = 90n + 90 \\
y = 3n + 86 & y = 2n + 90
\end{array}$$

$$3n + 86 = 2n + 90$$

$$n = 4$$

There were 4 previous exams this year.

PROBLEM 104 Mrs. Jones has a field in the shape of an equilateral triangle with a side of 40 feet. She ties a pet goat to a stake at one corner of the field. How long a rope must she use so that the goat can graze over one-half of the field?

Discussion The shaded area shown in Figure B-34 is the area over which the goat may graze. This area is in the form of a sector of a circle, with an angle of 60° and a radius equal to the length of the rope. The area of the equilateral triangle is found with the formula

$$A = \frac{s^2\sqrt{3}}{4}$$

$$A = \frac{40^2\sqrt{3}}{4}$$

$$A = \frac{1600\sqrt{3}}{4}$$

$$A = 400\sqrt{3}$$

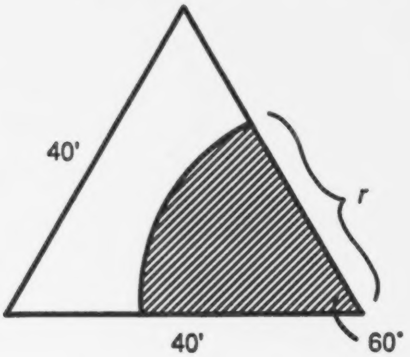


Figure B-34

Thus, $(\frac{1}{2})$(triangle) = (the sector)

$$\begin{aligned}
200\sqrt{3} &= \frac{60}{360} \cdot \pi r^2 \\
200\sqrt{3} &= \frac{1}{6}\pi r^2 \\
1200\sqrt{3} &= \pi r^2 \\
\frac{1200\sqrt{3}}{\pi} &= r^2 \\
\frac{2078.46}{\pi} &= r^2 \\
661.595 &= r^2 \\
25.72 &= r
\end{aligned}$$

She must use a rope that is approximately 25.7 feet.

PROBLEM 105 Two circles are concentric. The tangent to the inner circle forms a chord of 12 inches in the larger circle. Find the area of the "ring" between the two circles.

Discussion The drawing in Figure B-35 shows the situation. Represent the radii of the two circles by r_1 and r_2, respectively. Now radius OC extended to D is perpendicular to tangent (chord) ACB and bisects it at C.

$$\begin{aligned}
\text{Area} &= \pi r_2^2 - \pi r_1^2 \\
&= \pi(r_2^2 - r_1^2)
\end{aligned}$$

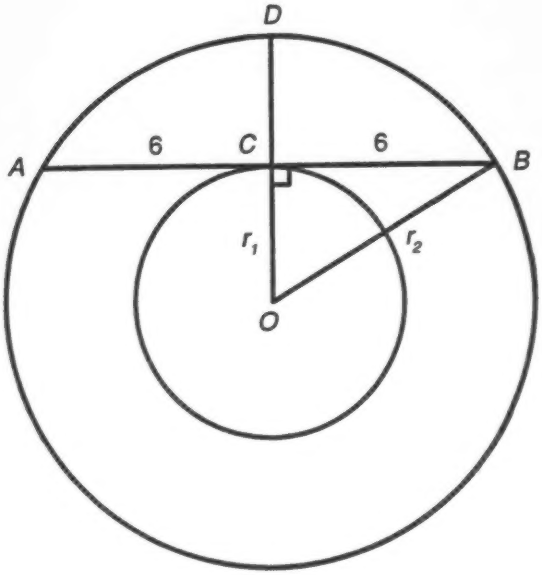


Figure B-35

But in right triangle *BOC*,

$$r_1^2 + 6^2 = r_2^2$$
$$36 = (r_2^2 - r_1^2)$$

Now substitute:

$$A = \pi \cdot 36$$

Notice that the regional area is independent of the actual radii of the two circles.

PROBLEM 106 The Appletree Garden Service must plant 69,489 apple trees in the 9 days before winter sets in. Every day after the first, the foreman puts 6 additional men on the job. However, every day after the first, each man plants 5 fewer trees than each did on the previous day. As a result, the number of trees planted per man keeps going down each day. What was the largest number of trees planted on any one day?

Discussion If we let the number of men working on the middle (the fifth) day be m, and the number of trees planted on that day by each man be n, then mn trees were planted on the fifth day. On the fourth day, there were $(m - 6)(n + 5)$ trees planted; on the sixth day, there were $(m + 6)(n - 5)$ trees planted; and so on. Thus the total for the 9 days would be

$$9mn - 1{,}800 = 69{,}489$$

or

$$mn = 7{,}921$$

Now 7,921 is the square of 89 (a prime number). Hence the chart of the work done would be as follows:

Day 1	65 men	×	109 trees	=	7,085
Day 2	71 men	×	104 trees	=	7,384
Day 3	77 men	×	99 trees	=	7,623
Day 4	83 men	×	94 trees	=	7,802
Day 5	89 men	×	89 trees	=	7,921
Day 6	95 men	×	84 trees	=	7,980
Day 7	101 men	×	79 trees	=	7,979
Day 8	107 men	×	74 trees	=	7,918
Day 9	113 men	×	69 trees	=	7,797
					69,489

The largest number of trees planted on any one day was 7,980 trees on the sixth day.

PROBLEM 107 A map of a local town is shown in Figure B-36. Billy lives at the corner of 4th Street and Fairfield Avenue. Betty lives

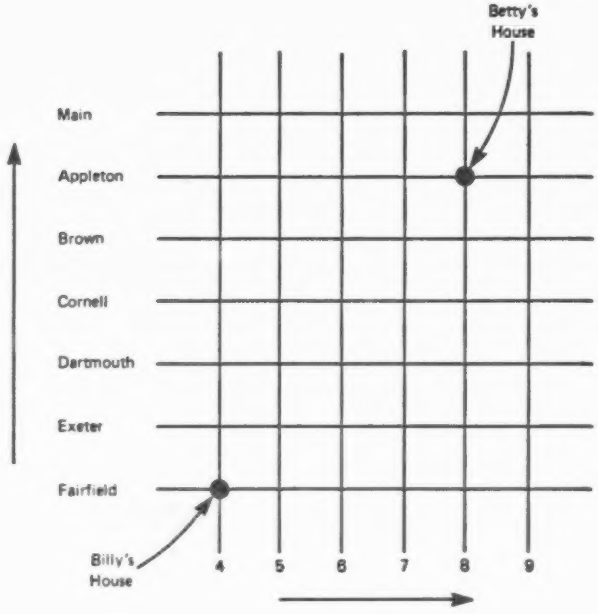

Betty's House
Main
Appleton
Brown
Cornell
Dartmouth
Exeter
Fairfield
Billy's House
4
5
6
7
8
9

Figure B-36

at the corner of 8th Street and Appleton Avenue. Billy decides that he will visit Betty once a day after school until he has traveled every different route to her house. Billy agrees to travel only east and north. How many different routes can Billy take to get to Betty's house?

Discussion The first attempts by many students to solve this problem usually involve trying to draw and count all of the different routes. This procedure is extremely cumbersome. It is easier to consider the number of different routes to each point on the grid in turn (Figure B-37). Notice that the numbers on the grid form the Pascal Triangle (Figure B-38).

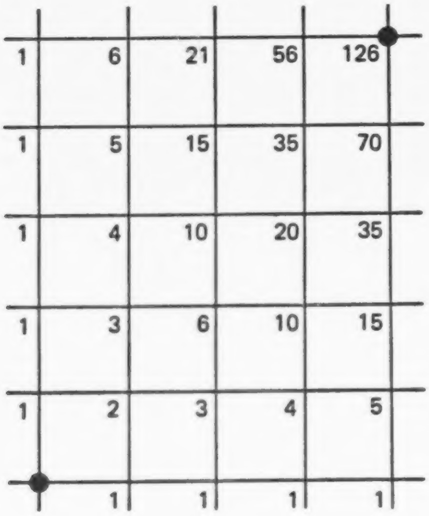


Figure B-37

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1

Figure B-38

PROBLEM 108 An interesting variation on Problem 107 is the following. Suppose we limit the direction of movement along the streets to that shown by the arrows in Figure B-39. How many different routes are there from point *A* to point *L*?

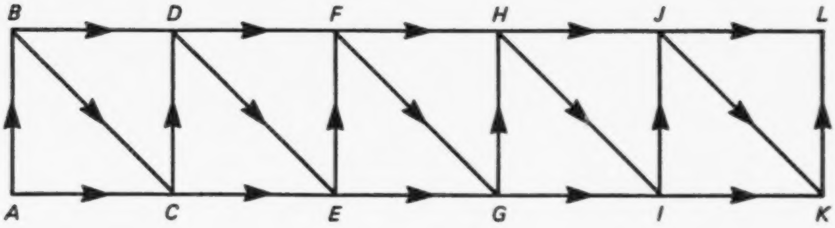


Figure B-39

Discussion If we again reduce the complexity of the problem and examine the number of possible routes to each of the different points on the grid, we find another interesting series of numbers. (See Figure B-40.)

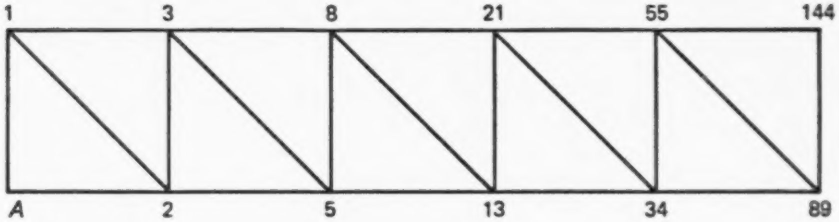


Figure B-40

The series 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, . . . is the Fibonacci sequence of numbers—that is, each term after the first two is derived by adding the preceding two terms of the series. Thus, there will be 144 different routes from *A* to *L*.

PROBLEM 109 The mathematician August DeMorgan wrote, "I was x years old in the year x^2." If DeMorgan was born prior to 1900 but lived into the twentieth century, in what year was he born?

Discussion We need the first perfect square greater than 1900. By "guess and test," along with a calculator, we find that 1936 is a perfect square. Thus $x^2 = 1936$ and $x = 44$. This means that DeMorgan was born in 1936 − 44, or 1892. (Notice that the next perfect square would be 45^2, or 2025. However, this would make his birth year 2025 − 45 = 1980, not prior to 1900.)

PROBLEM 110 Janice has a large cardboard box such that the area of one side of the box is 120 square feet. The area of a second side is 72 square feet, while the area of the third side is 60 square feet. What is the volume of the box?

Discussion In Figure B-41, the area of side $C = 120$; the area of side $B = 72$; and the area of side $A = 60$.

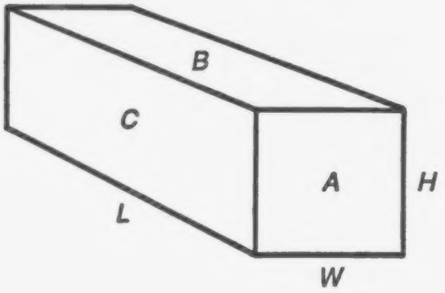


Figure B-41

Since the problem asks for the volume while giving the areas of the sides, we can solve for the volume *without* solving for the lengths of the individual sides.

$$V = L \cdot W \cdot H$$
$$V \cdot V = L \cdot W \cdot H \cdot L \cdot W \cdot H$$
$$V^2 = (LW)(WH)(LH)$$

But $LW = B$, $WH = A$, and $LH = C$. Thus,

$$V^2 = (72)(60)(120)$$
$$V^2 = 518{,}400$$
$$V = 720$$

Notice, too, that

$$V^2 = (72)(60)(120)$$
$$V^2 = (72)(60)(60)(2)$$
$$V^2 = (144)(60)(60)$$
$$V = (12)(60) = 720$$

and the calculations can be performed without pencil and paper.

PROBLEM 111 A steel band is tightly fitted around the Equator. The band is removed and cut, and an additional 10 feet is added. The band now fits more loosely than it did before. How high off the ground is the band?

Discussion Reduce the problem to that of a circle with radius r, so its circumference is represented by $2\pi r$. Now, when we increase the circumference by 10 feet, we increase the radius of the steel band by x (see Figure B-42).

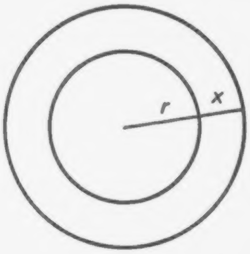


Figure B-42

Thus, we obtain the equation

$$2\pi(r + x) = 2\pi r + 10$$
$$2\pi r + 2\pi x = 2\pi r + 10$$
$$2\pi x = 10$$
$$x = \frac{10}{2\pi}$$

The steel band now fits approximately 1.6 feet above the surface of the earth.

PROBLEM 112 Each side of an equilateral triangle is 40 units long. A second triangle is inscribed by joining the midpoints of the sides of the original triangle. The process is continued, forming new triangles by joining the midpoints of the sides of the previous ones. Find the perimeter of the sixth triangle in the sequence.

Discussion Once again a drawing (or a set of drawings), together with a table, reveals the result. Students should recall the geometric fact that a line connecting the midpoints of two sides of a triangle is parallel to and equals one-half of the third side. Thus each triangle in the sequence is also equilateral, with a perimeter equal to one-half of the previous perimeter. (See Figure B-43.)

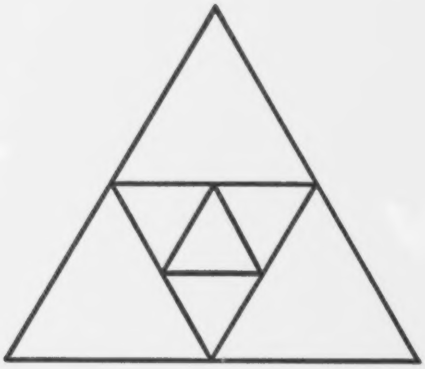


Figure B-43

	Side	*Perimeter*
First triangle	40	120
Second triangle	20	60
Third triangle	10	30
Fourth triangle	5	15
Fifth triangle	2.5	7.5
Sixth triangle	1.25	3.75

The sixth triangle has a perimeter of 3.75 units.

PROBLEM 113 If a man walks to work and rides back home, it takes him an hour and a half. When he rides both ways, it takes only

30 minutes. How long would it take him to make the round trip by walking both ways?

Discussion We can represent his rate when walking by x and his rate when riding by y. Let the distance each way be represented by D. Then

(a) When he walks to work and rides back:

$$\frac{D}{x} + \frac{D}{y} = 90$$

$$\frac{D}{x} + 15 = 90$$

$$\frac{D}{x} = 75$$

(b) When he rides both ways:

$$\frac{D}{y} + \frac{D}{y} = 30$$

$$\boxed{\frac{D}{y} = 15}$$

To walk one way takes 75 minutes, or an hour and a quarter. The round trip would take him 2½ hours.

PROBLEM 114 The total number of angles in two regular polygons is 13, while the total number of diagonals is 25. How many angles are there in each of the polygons?

Discussion At first, particularly to a student of algebra, this would appear to be a routine problem in which the student would let x_1 = the number of angles in the first polygon and x_2 = the number of angles in the second polygon, etc. However, this approach to a solution soon leads nowhere, and a more careful analysis leads to the making of drawings and the use of the "guess and test" strategy.
Since the number of angles and sides of any polygon is the same, we are searching for two regular polygons the sum of whose sides is 13. Thus, 3 and 10, 4 and 9, 5 and 8, etc., should be examined. A series of drawings to show the relationship between the number of sides and the number of diagonals reveals the answer to be a pentagon and an octagon.

PROBLEM 115 A woman was ⅜ of the way across a bridge when she heard the Wabash Cannonball Express approaching the bridge at 60 miles per hour. She quickly calculated that she could *just* save herself by running to either end of the bridge at top speed. How fast could she run?

Discussion Draw a diagram as shown in Figure B-44. We see that there are two different situations to consider.

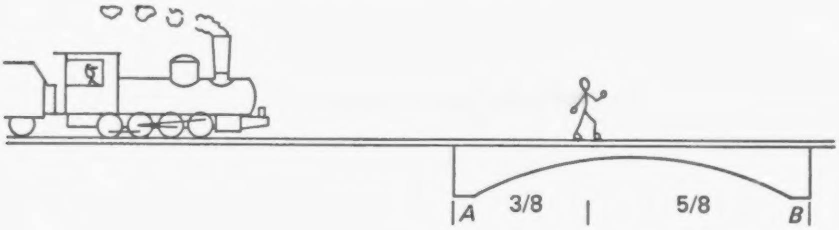


Figure B-44

If she can *just* get to either end of the bridge before the train arrives at that end, let her run away from the train towards point B. When the train arrives at point A, she will have covered an additional $\frac{3}{8}$ of the bridge's length (the distance to A), placing her $\frac{3}{4}$ of the way across the bridge. She can now run the remaining $\frac{1}{4}$ of the bridge in the same time it will take the train to cross the entire $\frac{4}{4}$ of the bridge. Thus her rate is $\frac{1}{4}$ that of the train, or 15 miles per hour.

Notice that a more formal algebraic solution can also be used. Let the bridge be some convenient length, say 8 units. Let x represent the distance the train is from point A; let y represent the woman's rate. Then we obtain the following two equations:

$$\frac{3}{y} = \frac{x}{60} \quad \text{(if the woman runs towards point } A\text{)}$$

$$\frac{5}{y} = \frac{x+8}{60} \quad \text{(if the woman runs towards point } B\text{)}$$

from which we get $y = 15$.

PROBLEM 116 Irene and Allan each drive a small car. Irene averages 40 miles per gallon with her car, while Allan averages 30 miles per gallon with his. They both attend the same college and live the same distance from the campus. If Allan needs 1 gallon more of gasoline than Irene does for 5 round trips to campus, how far does each live from the campus?

Discussion An algebraic representation can simulate the action. Let D = the distance each lives from campus. Then their respective fuel consumption for 5 round trips is:

$$\text{Irene} = \frac{10D}{40} = \frac{D}{4}$$

$$\text{Allan} = \frac{10D}{30} = \frac{D}{3}$$

Therefore,

$$\frac{D}{3} - \frac{D}{4} = 1$$

$$4D - 3D = 12$$

$$D = 12$$

They each live 12 miles from the campus.

PROBLEM 117 Two jumping ants start at the origin. One travels along the positive half of the x-axis, while the other travels along the positive half of the y-axis. Each ant jumps one unit on its first jump, $\frac{1}{2}$ unit on its second jump, $\frac{1}{4}$ unit on its third jump, and so on, where on any subsequent jump, the ant jumps only half as far as on the immediately preceding jump. How far apart will the ants be, assuming that their

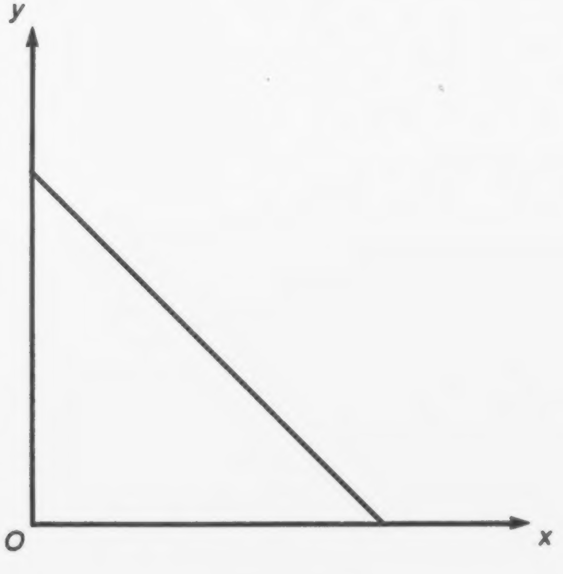


Figure B-45

moves are frequent and take place over an infinite amount of time?

Discussion Once again, a drawing helps to visualize the action (see Figure B-45). Since the axes are perpendicular and each ant's motion is the same, a geometric interpretation reveals an isosceles right triangle in which the hypotenuse is the required answer. Each ant travels a distance given by the infinite geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$. Thus $a = 1$ and $r = \frac{1}{2}$.

$$S = \frac{a}{1 - r} = \frac{1}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

and the hypotenuse is $2\sqrt{2}$.

PROBLEM 118 An immortal spider starts at *A* and crawls along a perpendicular to radius *OB* until it reaches point C. (See Figure B-46.) Then it crawls along a perpendicular to radius *OA* to point *D*; then along a perpendicular to *OB* to point *E*, and so on, ad infinitum. Find the distance traveled by the spider if angle *AOB* contains 30° and radius $OA = 1$ unit.

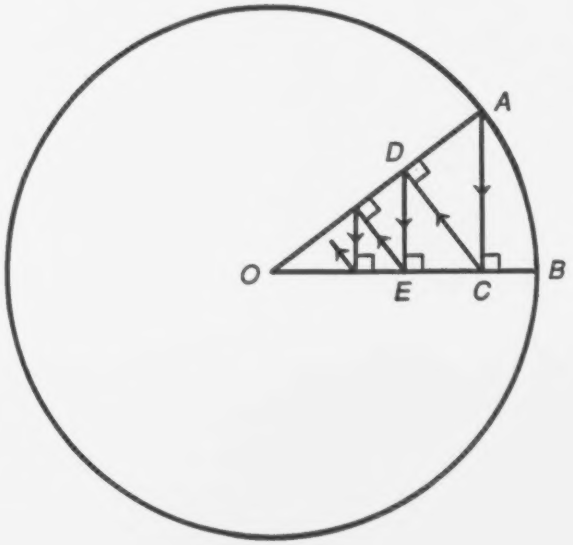


Figure B-46

Discussion The distance the spider moves is found by finding the lengths of the sides of the various 30°–60°–90° right triangles. The sequence needed is:

$$\overline{AC} + \overline{CD} + \overline{DE} + \overline{ED} + \cdots$$

$$\frac{1}{2} + \frac{\sqrt{3}}{4} + \frac{3}{8} + \frac{3\sqrt{3}}{16} + \cdots \frac{3^{n-1/2}}{2^n}$$

This is an infinite geometric series, in which r is found by dividing any term by the term preceding it:

$$r = \frac{\sqrt{3}}{4} \div \frac{1}{2} = \frac{\sqrt{3}}{2}$$

Then,

$$S = \frac{a}{1-r} = \frac{\frac{1}{2}}{1 - \frac{\sqrt{3}}{2}} = \frac{1}{2 - \sqrt{3}} = 2 + \sqrt{3}$$

PROBLEM 119 At a country fair, a game is often played in which the contestants toss a coin onto a large table that has been ruled into squares of the same size. If the coin lands *entirely* within a square, the contestant wins a prize. Otherwise, the player loses the coin that he or she has tossed. If the squares on a table are 25 mm on a side and Annie's coin has a radius of 10 mm, what is the probability that Annie will win a prize?

Discussion Figure B-47 shows a section of the table with 16 squares. Let's consider one square, *ABCD*, anywhere on the grid. Since the squares are all congruent, it follows that the probability of winning over the entire grid is proportional to the probability of success on any one square. Now, a success on square *ABCD* takes place when the center of the coin (a circle with radius = 10 mm) lies no closer to a side of the square than its radius. If it is closer to a side than its radius, it will touch the side and the player will lose. Thus, we construct a smaller square *EFGH* inside *ABCD* such that each side of *EFGH* is exactly 10 mm from the sides of *ABCD*. To win, the center of the coin must fall within *EFGH*. Thus the probability of winning is the ratio of the area of *EFGH* to the area of *ABCD*. Since the radius of the coin is 10 mm, the sides of *EFGH* are each 5 mm (25 − 10 − 10). Thus the ratio of the areas is

$$\frac{5^2}{25^2} = \frac{1}{25}$$

Thus the probability that Annie will win is $\frac{1}{25}$.

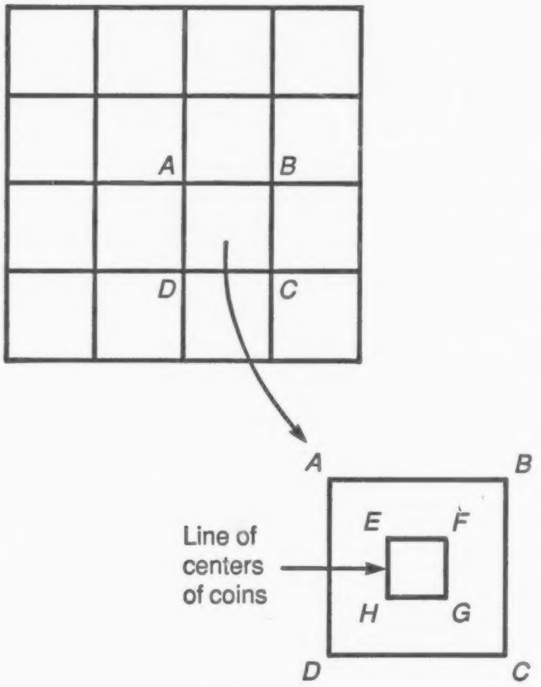


Figure B-47

PROBLEM 120 A standard deck of playing cards contains 26 black cards and 26 red cards, or 52 cards in all. A deck is randomly divided into two unequal piles, such that the probability of drawing a red card from the smaller pile is $\frac{1}{3}$. At the same time, the probability of drawing a black card from the larger pile is $\frac{5}{14}$. How many cards are in the larger pile?

Discussion The number of black cards in the larger pile must be a multiple of 14, since the probability of drawing a black card is $\frac{5}{14}$. Thus there will be 14, 28, or 42 cards in this pile. We can eliminate 14, since this would leave 38 cards for the other (or smaller) pile. If we assume that there are 42 cards, this makes 15 of them black ($\frac{5}{14}$ of 42). This leaves $42 - 15$ or 27 red cards in the pack—but the deck only has 26 red cards to begin with. Now, let's examine 28 cards. This implies that 10 are black ($\frac{5}{14}$ of $28 = 10$). This leaves 18 red in this pack, and 8 left over for the smaller pack. Since $\frac{8}{24}$ does equal $\frac{1}{3}$, this is a valid solution. Thus there are 28 cards in the larger pack.

PROBLEM 121 A wealthy philanthropist sets aside a certain amount of money to be distributed equally among the needy at a shelter each week. One week he remarked, "If there are five fewer of you here next week, each will receive two dollars more." Unfortunately, instead of being fewer, there were actually four more people present the following week, and each received one dollar less. How much did the philanthropist set aside each week?

Discussion This complicated scenario can be simplified by a set of three algebraic equations:

Let x = the number of people present the first time
y = the amount each person received the first time
n = the amount the philanthropist had set aside

Then,

$$\begin{aligned} xy &= n \\ (x - 5)(y + 2) &= n \\ (x + 4)(y - 1) &= n \end{aligned}$$

From the first two equations, we obtain

$$\begin{aligned} xy - 5y + 2x - 10 &= xy \\ 2x &= 5y + 10 \\ x &= \frac{5y + 10}{2} \end{aligned}$$

Now we substitute in the third equation,

$$\begin{aligned} xy + 4y - x - 4 &= xy \\ 4y - 4 &= x \\ 4y - 4 &= \frac{5y + 10}{2} \\ 8y - 8 &= 5y + 10 \\ 3y &= 18 \\ y &= 6 \\ x &= 20 \end{aligned}$$

The philanthropist had set aside 6 × 20, or \$120, for distribution each week.

SECTION C

A Bibliography of Problem-Solving Resources

A Bibliography of Problem-Solving Resources

Adler, Irving. *Magic House of Numbers*. John Day Company, New York, 1974.

Arithmetic Teacher. The entire November 1977 and February 1982 issues are devoted to problem solving. National Council of Teachers of Mathematics, Reston, Virginia.

Ball, W. W. R. *Mathematical Recreations and Essays* (12th edition, revised by H. S. M. Coxeter). Reprinted by the University of Toronto Press, Toronto, 1974.

Barnard, Douglas. A Book of Mathematical and Reasoning Problems. D. Van Nostrand Company, New York, 1962.

Bransford, John D., and Stein, Barry S. *The Ideal Problem Solver*. W. H. Freeman and Company, San Francisco, 1984.

Brown, Stephen, and Walter, Marion. *The Art of Problem Posing*. Franklin Institute Press, Philadelphia, 1983.

Butts, Thomas. *Problem Solving in Mathematics*. Scott Foresman and Company, Glenview, Illinois, 1973.

Charosh, Mannis (editor). *Mathematical Challenges*. National Council of Teachers of Mathematics, Reston, Virginia, 1973.

Dodson, J. *Characteristics of Successful Insightful Problem Solvers*. University Microfilm, Number 71-13,0048, Ann Arbor, Michigan, 1970.

Dolan, Daniel T., and Williamson, James. *Teaching Problem Solving Strategies*. Addison-Wesley Publishing Company, Menlo Park, California, 1983.

Dudeney, H. E. *Amusements in Mathematics*. Dover Publishing Company, New York, 1958.

Fisher, Lyle, and Kennedy, Bill. *Brother Alfred Brousseau Problem-Solving and Mathematics Competition*. Dale Seymour Publications, Palo Alto, California, 1984.

Fixx, James. *Games for the Superintelligent*. Doubleday and Company, Garden City, New York, 1972.

Frolichstein, Jack. *Mathematical Fun, Games and Puzzles*. Dover Publishing Company, New York, 1967.

Gardner, Martin. *Aha!* Scientific American/W. H. Freeman and Company, San Francisco, California, 1978.

________. *Gotcha!* Scientific American/W. H. Freeman and Company, San Francisco, California, 1982.

________. *Scientific American Book of Mathematical Puzzles and Diversions*. Simon and Schuster, New York, 1959.

________. *Second Scientific American Book of Mathematical Puzzles and Diversions*. Simon and Schuster, New York, 1961.

Greenes, Carol, Gregory, John, and Seymour, Dale. *Successful Problem Solving Techniques*. Creative Publications, Palo Alto, California, 1978.

Greenes, Carol, Spunkin, Rika, and Dombrowski, Justine M. *Problem-Matics*. Creative Publications, Palo Alto, California, 1981.

Haynes, John R. *The Complete Problem Solver*. Franklin Institute Press, Philadelphia, 1981.

Heafford, Philip. *The Math Entertainer*. Harper and Row Publishers, New York, 1973.

Hirsch, Thomas L., and Wylie, C. Ray. *The Problem Pockets—Critical Thinking Activities*. Creative Publications, Palo Alto, California, 1986.

Holden, Linda. *Thinker Tasks: Critical Thinking Activities*. Creative Publications, Palo Alto, California, 1986.

Jacoby, Oswald, and Benson, William H. *Mathematics for Pleasure*. Fawcett World Library, New York, 1965.

Kaufman, Gerald L. *The Book of Modern Puzzles*. Dover Publishing Company, New York, 1954.

Kraitchik, Maurice. *Mathematical Recreations* (2nd edition). Dover Publishing Company, New York, 1972.

Krulik, Stephen, and Rudnick, Jesse A. *A Sourcebook for Teaching Problem Solving*. Allyn and Bacon, Inc. Newton, Massachusetts, 1984.

________. *Problem Solving in Math* (Levels G and H). Scholastic Book Services, New York, 1982.

Lane County Mathematics Project. *Problem Solving in Mathematics* (Grade 9). Dale Seymour Publications, Palo Alto, California, 1983.

Lenchner, George. *Creative Problem Solving in School Mathematics*. Houghton Mifflin Company, Boston, 1983.

Longley-Cook, L. H. *New Math Puzzle Book*. Van Nostrand Reinhold Company, New York, 1970.

May, Francis B. *Introduction to Games of Strategy*. Allyn and Bacon, Inc., Newton, Massachusetts, 1970.

Mira, Julio. *Mathematical Teasers*. Barnes and Noble, New York, 1970.

Mott-Smith, Geoffrey. *Mathematical Puzzles for Beginners and Enthusiasts* (2nd edition). Dover Publishing Company, New York, 1954.

Peck, Lyman. *Secret Codes, Remainder Arithmetic and Matrices*. National Council of Teachers of Mathematics, Reston, Virginia, 1961.

Pedersen, Jean J., and Armbruster, Franz O. *A New Twist: Developing Arithmetic Skills Through Problem Solving*. Addison-Wesley Publishing Company, Menlo Park, California, 1979.

Polya, George. *How To Solve It*. Princeton University Press, Princeton, New Jersey, 1971.

________. *Mathematical Discovery: On Understanding, Learning and Teaching Problem Solving* (2 volumes). John Wiley and Sons, New York, (Volume 1) 1962, (Volume 2) 1965.

________, and Kilpatrick, Jeremy. *The Stanford Mathematics Problem Book*. Teachers College Press, New York, 1971.

Problem Solving in School Mathematics (1980 Yearbook of National Council of Teachers of Mathematics). Reston, Virginia, 1980.

Ranucci, Ernest. *Puzzles, Problems, Posers and Pastimes* (3 volumes). Houghton Mifflin Company, Boston, 1972.

Reeves, Charles A. *Problem-Solving Techniques Helpful in Mathematics and Science*. National Council of Teachers of Mathematics, Reston, Virginia, 1987.

School, Science and Mathematics. The entire March 1978 issue is devoted to problem solving. School, Science and Mathematics Association, Kalamazoo, Michigan.

Shklarsky, D. O., Chentzov, N. N., and Yaglom, I. M. *The U.S.S.R. Olympiad Problem Book*. W. H. Freeman and Company, San Francisco, 1962.

Silver, Edward A. (editor). *Teaching and Learning Mathematical Problem Solving*. Lawrence Erlbaum Associates, Hillsdale, New Jersey, 1985.

Stacey, Kaye, and Southwall, Beth. *Teacher Tactics for Problem Solving*. Curriculum Development Centre, Canberra, Australia, 1983.

Throop, Sara. *Problem Solving* (3 volumes). Gamco Industries, Big Spring, Texas, 1983.

Wickelgren, Wayne. *How To Solve Problems*. W. H. Freeman and Company, San Francisco, 1974.

Williams, J. D. *The Compleat Strategyst* (revised edition). McGraw-Hill Book Company, New York, 1965.

Wylie, C. R. *One Hundred Puzzles in Thought and Logic*. Dover Publishing Company, New York, 1957.

SECTION D

Masters for Selected Problems

NAME ______________________________ DATE ______________

Problem: The new school has exactly 1,000 lockers and exactly 1,000 students. On the first day of school, the students meet outside the building and agree on the following plan: The first student will enter the school and open all of the lockers. The second student will then enter the school and close every locker with an even number (2, 4, 6, 8, . . .). The third student will then "reverse" every third locker. That is, if the locker is closed, he or she will open it; if the locker is open, he or she will close it. The fourth student will reverse every fourth locker, and so on until all 1,000 students in turn have entered the building and reversed the proper lockers. Which lockers will finally remain open?

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NAME ______________________________ **DATE** __________________

Problem: How many squares are there on a standard checkerboard?

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NAME ______________________________ DATE ____________________

Problem: Jane and Susan are the first two people in line to buy their tickets to the rock concert. They started the line at 6:00 A.M. Every 20 minutes, 3 more people than are in the line at that time arrive and join the line. How many people will be in the line when the tickets go on sale at 9:00 A.M.?

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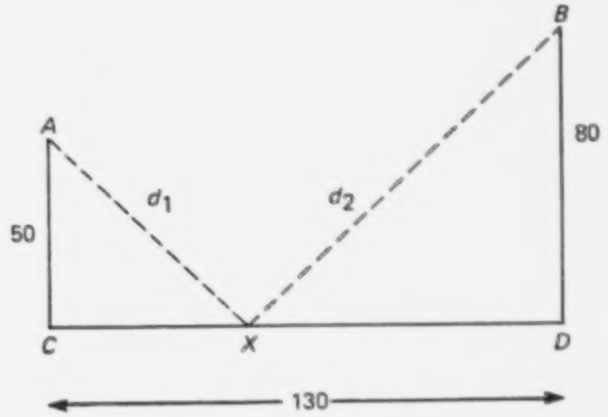
NAME ____________________ DATE ____________

Problem: A pirate ship at point *A* in the diagram shown is 50 meters directly north of point C on the shore. Point *D*, also on the shore and due east of point C, is 130 meters from point C. Point *B*, a lighthouse, is due north of point *D* and 80 meters from point *D*. The pirate ship must touch the shoreline and then sail to the lighthouse. Find the location of point X on the shoreline, so that the path from *A* to X to *B* will be a minimum.



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NAME ______________________________ DATE ____________________

Problem: In an office, there are two square windows. Each window is 4 feet high, yet one window has an area that is twice that of the other window. Explain how this can be true.

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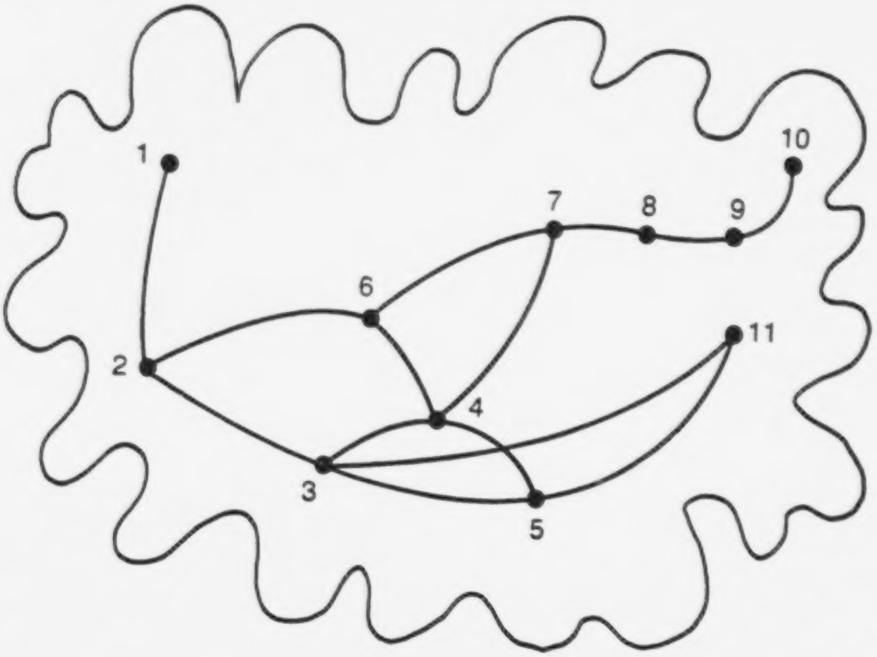
NAME ______________________________ DATE __________________

Problem: In the outer reaches of space, there are eleven relay stations for the Intergalactic Space Ship Line. There are space ship routes between the relay stations as shown in the map. Eleven astronauts have been selected as communications operators, one for each station. They are Alex, Barbara, Cindy, Donna, Elvis, Frances, Gloria, Hal, Irene, Johnny, and Karl. The two people in stations with connecting routes will be talking to each other a great deal, to discuss space ships that fly from station to station. It would be helpful if these people were friendly to each other. Here are the pairs of people who are friends:

Alex–Barbara
Gloria–Johnny
Donna–Irene
Cindy–Hal
Johnny–Cindy
Hal–Frances
Gloria–Irene
Alex–Gloria
Alex–Donna
Donna–Karl
Irene–Karl
Donna–Elvis
Karl–Elvis
Johnny–Irene

Place the eleven people in the eleven stations so that the people on connecting stations are friends.



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NAME ______________________________ DATE ________________

Problem: The following advertisement appeared in the real estate section of a local newspaper:

INVESTMENT FOR THE FUTURE!
LAND FOR SALE–
ONLY 5¢ PER SQUARE INCH!!

Invest your money now in land,
in an area that is soon to be developed.
For information, call or write

Land Developers, Inc.

If you were to buy an acre of land as advertised, what amount would you be required to pay?

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NAME ________________________ DATE ______________

Problem: Two elevators each leave the sixth floor of a building at exactly 3:00 P.M. The first elevator takes 1 minute between floors, while the second elevator takes 2 minutes between floors. However, whichever elevator arrives at a given floor first must wait 3 minutes before leaving. Which elevator arrives at the ground floor first?

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NAME ______________________________ DATE ____________________

Problem: Two bicycle riders, Jeff and Nancy, are 25 miles apart, riding toward each other at speeds of 15 miles per hour and 10 miles per hour, respectively. A fly starts from Jeff and flies toward Nancy and then back to Jeff again and so on. The fly continues flying back and forth at a constant rate of 40 miles per hour, until the bicycle riders "collide" and crush the fly. How far has the fly traveled?

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NAME ______________________________ DATE ____________________

Problem: A family tree for a male bee is very unusual. A male bee has only one parent (a mother), while a female bee has two parents (a mother and a father). How many ancestors does a single male bee have, if we go back for 6 generations?

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NAME ______________________________ DATE ___________________

Problem: A paperboy agrees to deliver newspapers for one year. In return, he will receive a salary of $240 plus a new bike. He quits after 7 months, and receives $100 and the bike. What is the value of the bike?

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NAME ______________________________ DATE __________________

Problem: At a large picnic, there were 45 dishes served altogether. Every 3 people shared a dish of cole slaw between them. Every 4 people shared a dish of potato salad between them. Every 6 people shared a dish of hot dogs between them. How many people were at the picnic?

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NAME ______________________________ **DATE** ____________________

Problem: A travel agency offers a charter trip to Yellowstone National Park, charging $300 per person if all 150 places can be filled. If not, the price per ticket is increased by $5 for every place not sold. How many tickets should be sold to give the agency the maximum income for the trip?

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NAME ______________________ DATE ______________

Problem: Mr. Lopez, who is 6 feet tall, wants to install a mirror on his bedroom wall that will enable him to see a full view of himself. What is the minimum-length mirror that will serve his needs, and how should it be placed on his wall?

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NAME ______________________________ DATE ________________

Problem: Three cylindrical oil drums of 2-foot diameter are to be securely fastened in the form of a triangle by a steel band. What length of band will be required?

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NAME ______________________________ DATE __________________

Problem: A, B, and C decide to play a game of cards. They agree on the following procedure: When a player loses a game, he or she will double the amount of money that each of the other players already has. First A loses a hand and doubles the amount of money that B and C each have. Then B loses a hand and doubles the amount of money that A and C each have. Then C loses a hand and doubles the amount of money that A and B each have. The three players then decide to quit, and they find that each player now has $8. Who was the biggest loser?

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NAME ______________________ DATE ______________

Problem: During the recent census, a man told the census taker that he had three children. When asked their ages, he replied, "The product of their ages is 72. The sum of their ages is the same as my house number." The census taker ran to the door and looked at the house number. "I still can't tell," she complained. The man replied, "Oh, that's right. I forgot to tell you that the oldest one likes chocolate pudding." The census taker promptly wrote down the ages of the three children. How old are they?

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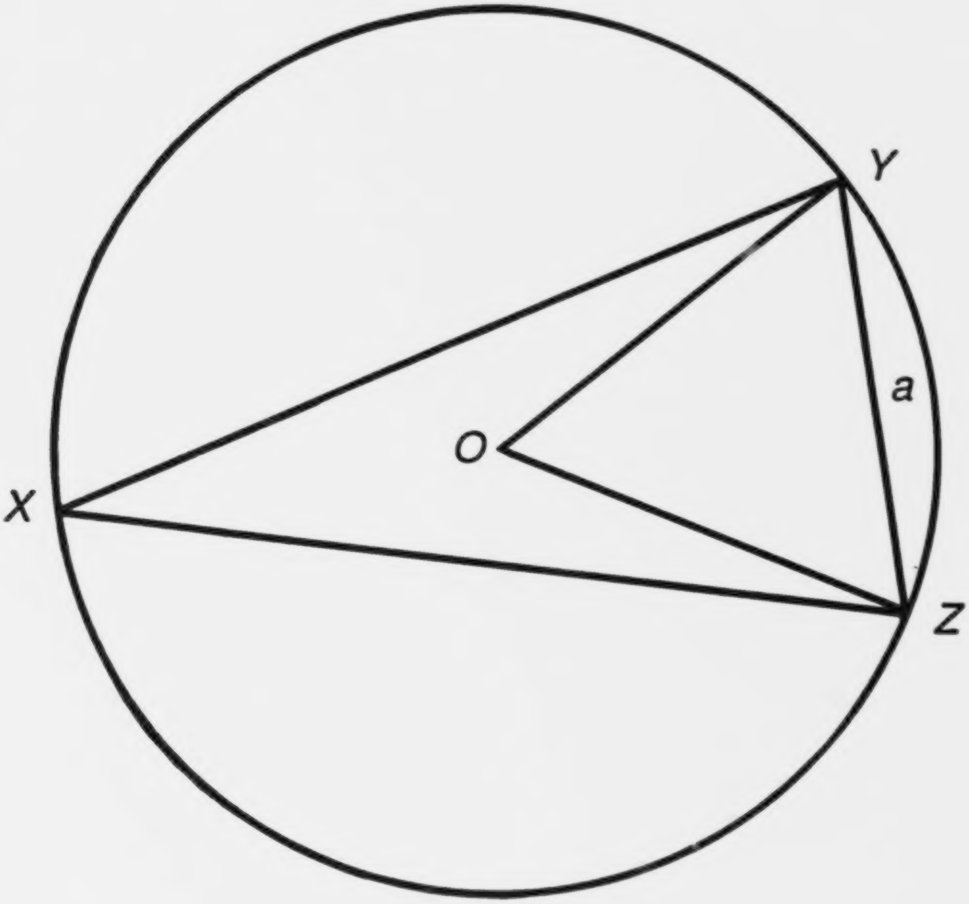
NAME ______________________________ DATE ________________

Problem: In circle O, a diameter $= 2a$ and chord $\overline{YZ} = a$. Find the measure of angle YXZ.



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NAME ______________________________ DATE ____________________

Problem: A cake in the form of a cube falls into a large vat of frosting and comes out frosted on all 6 faces. The cake is then cut into smaller cubes, each 1 inch on an edge. The cake is cut so that the number of pieces with frosting on 3 faces will be $\frac{1}{8}$ the number of pieces having no frosting at all. We wish to have exactly enough pieces of cake for everyone. How many people will receive pieces of cake with frosting on exactly 3 faces? On exactly 2 faces? On exactly 1 face? On no faces? How large was the original cake?

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NAME ______________________________ DATE __________________

Problem: The probability of rolling a 2 on a standard pair of dice is $\frac{1}{36}$; the probability of rolling a 3 is $\frac{2}{36}$ (a 1-2 or a 2-1); and so on. How could you remark a pair of dice so that the probability of throwing each number from 1 through 12 was the same?

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NAME ______________________________ DATE __________________

Problem: A bridge that spans a bay is 1 mile long and is suspended from two supports, one at each end. As a result, when it expands a total of 2 feet from the summer heat, it "buckles" in the center, causing a bulge. How high is the bulge?

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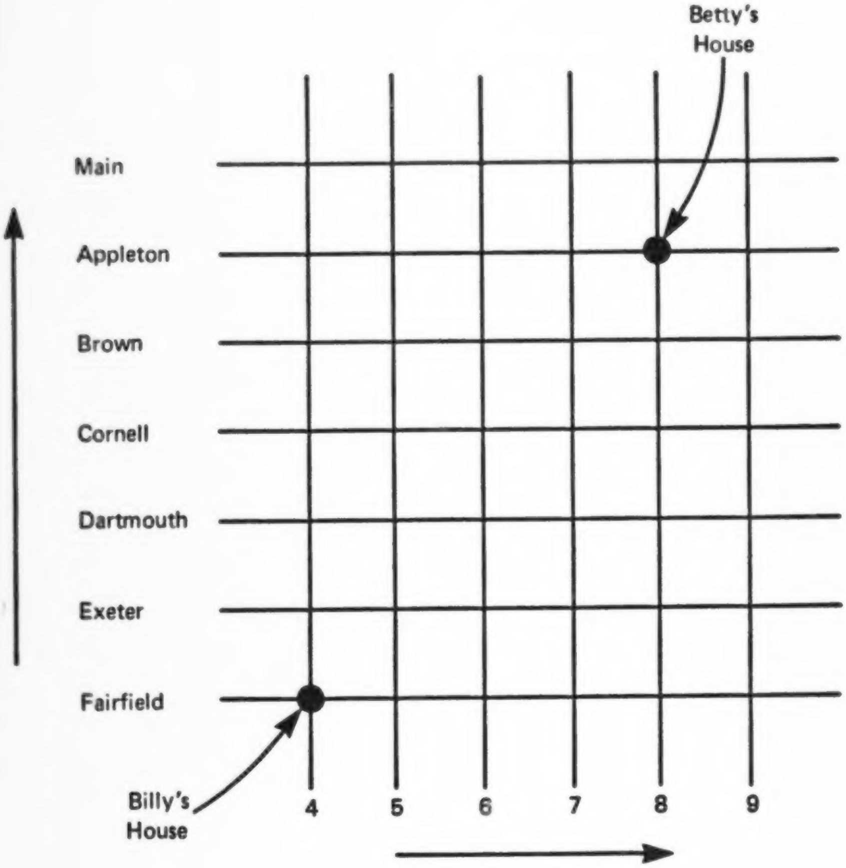
NAME ______________________________ DATE ________________

Problem: A map of a local town is shown in the figure below. Billy lives at the corner of 4th Street and Fairfield Avenue. Betty lives at the corner of 8th Street and Appleton Avenue. Billy decides that he will visit Betty once a day after school until he has tried every different route to her house. Billy agrees to travel only east and north. How many different routes can Billy take to get to Betty's house?


Betty's House
Main
Appleton
Brown
Cornell
Dartmouth
Exeter
Fairfield
Billy's House
4
5
6
7
8
9

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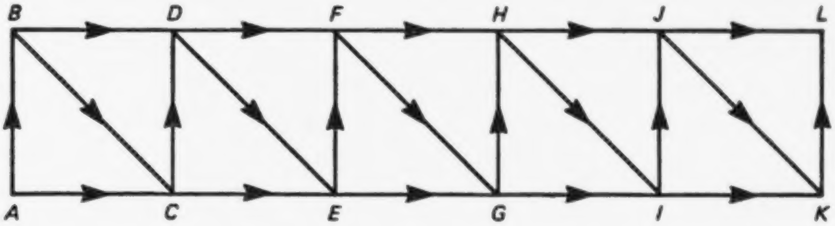
NAME ______________________ DATE ______________

Problem: A taxicab restricts its travel along the streets shown in the figure and in the directions shown by the arrows. How many different ways are there for the taxi to get from *A* to *L*?



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NAME ______________________________ DATE ________________

Problem: Janice has a large cardboard box such that the area of one side of the box is 120 square feet. The area of a second side is 72 square feet, while the area of the third side is 60 square feet. What is the volume of the box?

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NAME ______________________________ DATE ____________________

Problem: A steel band is tightly fitted around the Equator. The band is removed and cut, and an additional 10 feet is added. The band now fits more loosely than it did before. How high off the ground is the band?

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NAME ______________________________ DATE ____________________

Problem: Irene and Allan each drive a small car. Irene averages 40 miles per gallon with her car, while Allan averages 30 miles per gallon with his. They both attend the same college and live the same distance from the campus. If Allan needs 1 gallon more gasoline than Irene does for 5 round trips to campus, how far does each live from the campus?

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NAME ______________________________ **DATE** ____________________

Problem: Two jumping ants start at the origin. One travels along the positive half of the x-axis, while the other travels along the positive half of the y-axis. Each ant jumps one unit on its first jump, $\frac{1}{2}$ unit on its second jump, $\frac{1}{4}$ unit on its third jump, and so on. On any subsequent jump, the ant jumps only half as far as on the immediately preceding jump. How far apart will the ants be, assuming that their moves are frequent and take place over an infinite amount of time?

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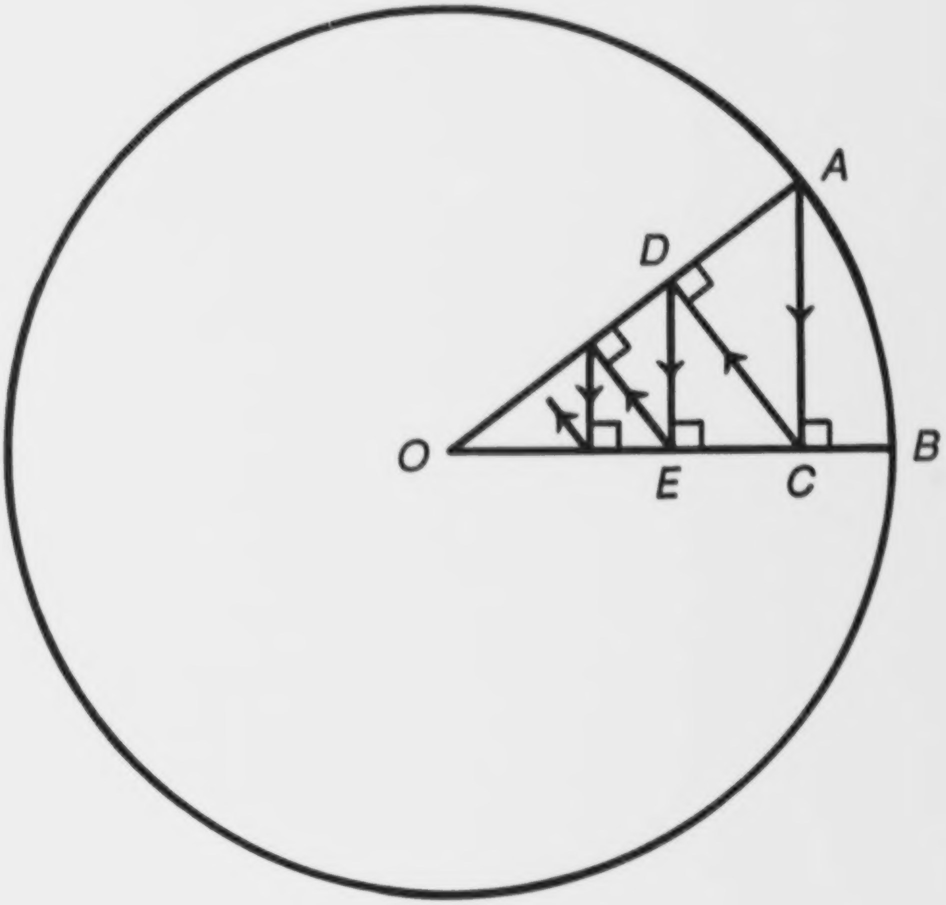
NAME ______________________________ DATE __________________

Problem: An immortal spider starts at *A* and crawls along a perpendicular to radius *OB* until it reaches point *C*. Then it crawls along a perpendicular to radius *OA* to point *D*; then along a perpendicular to *OB* to point *E*, and so on, ad infinitum. Find the distance traveled by the spider if angle *AOB* contains 30° and radius *OA* = 1 unit.



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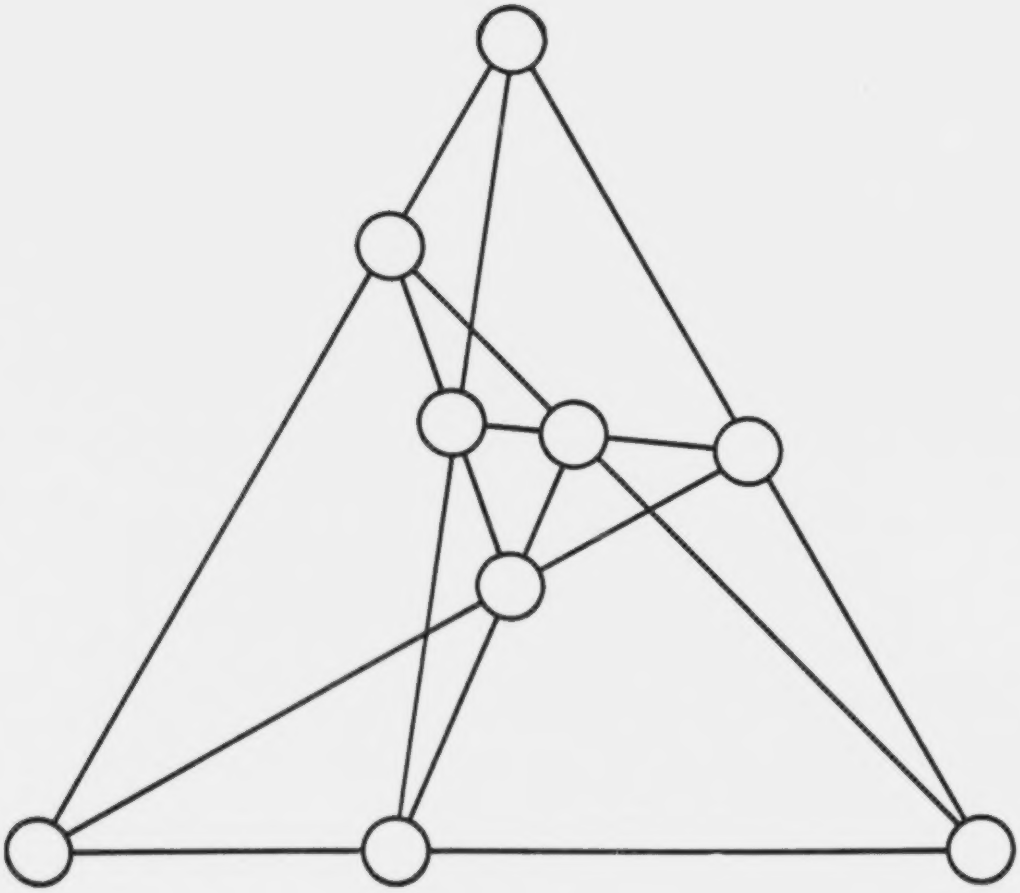
SECTION E

Masters for Strategy Game Boards



Triangular Tic-Tac-Toe

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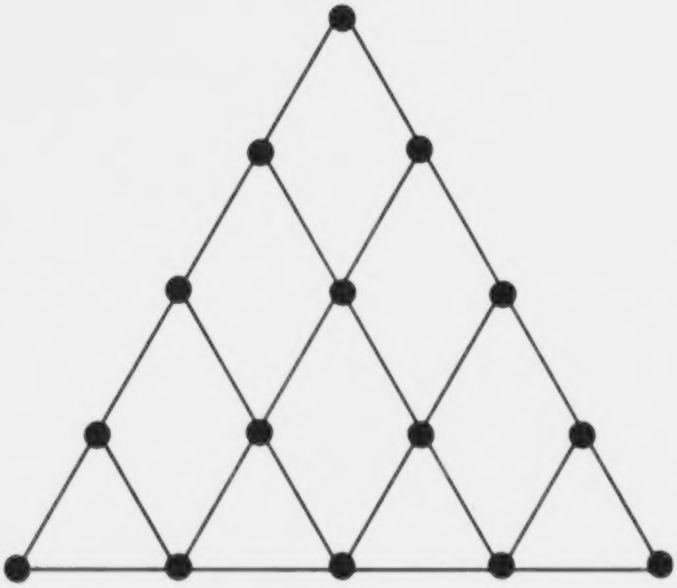

Dots-in-a-Row Tic-Tac-Toe

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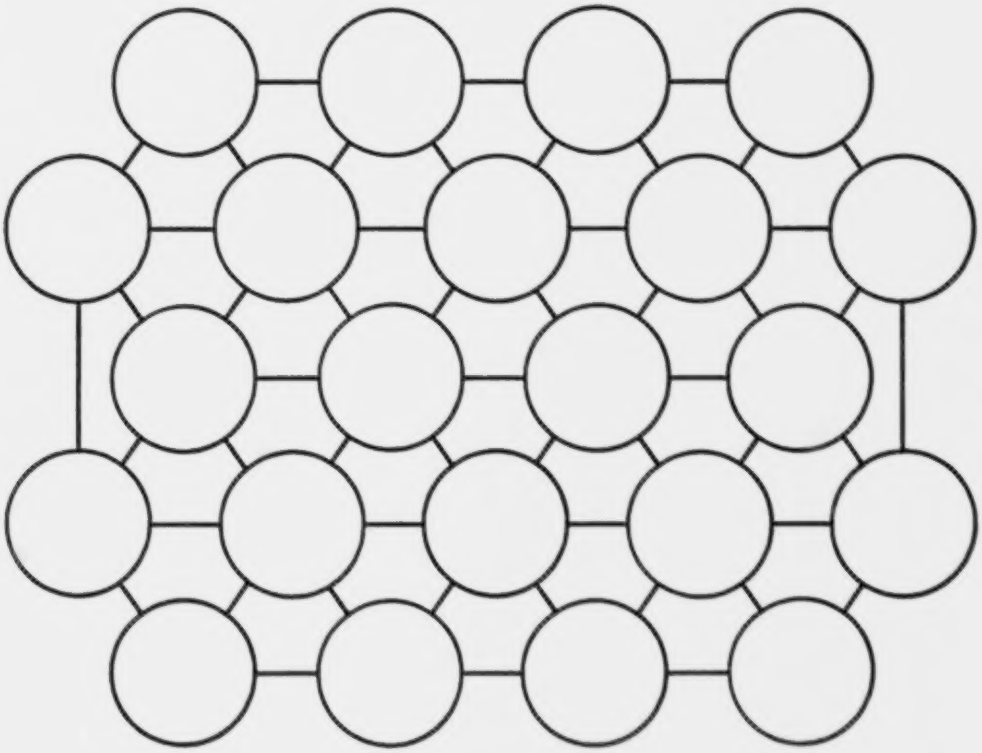

Tac-Tic-Toe, Chinese Version

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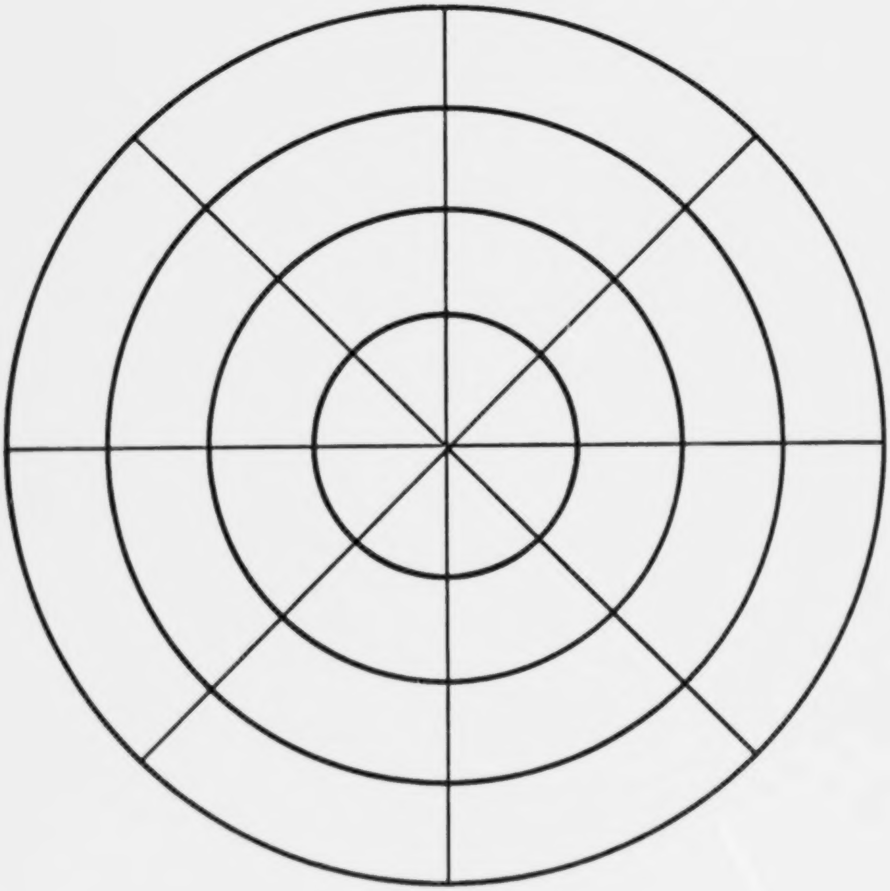

Spiral Tic-Tac-Toe

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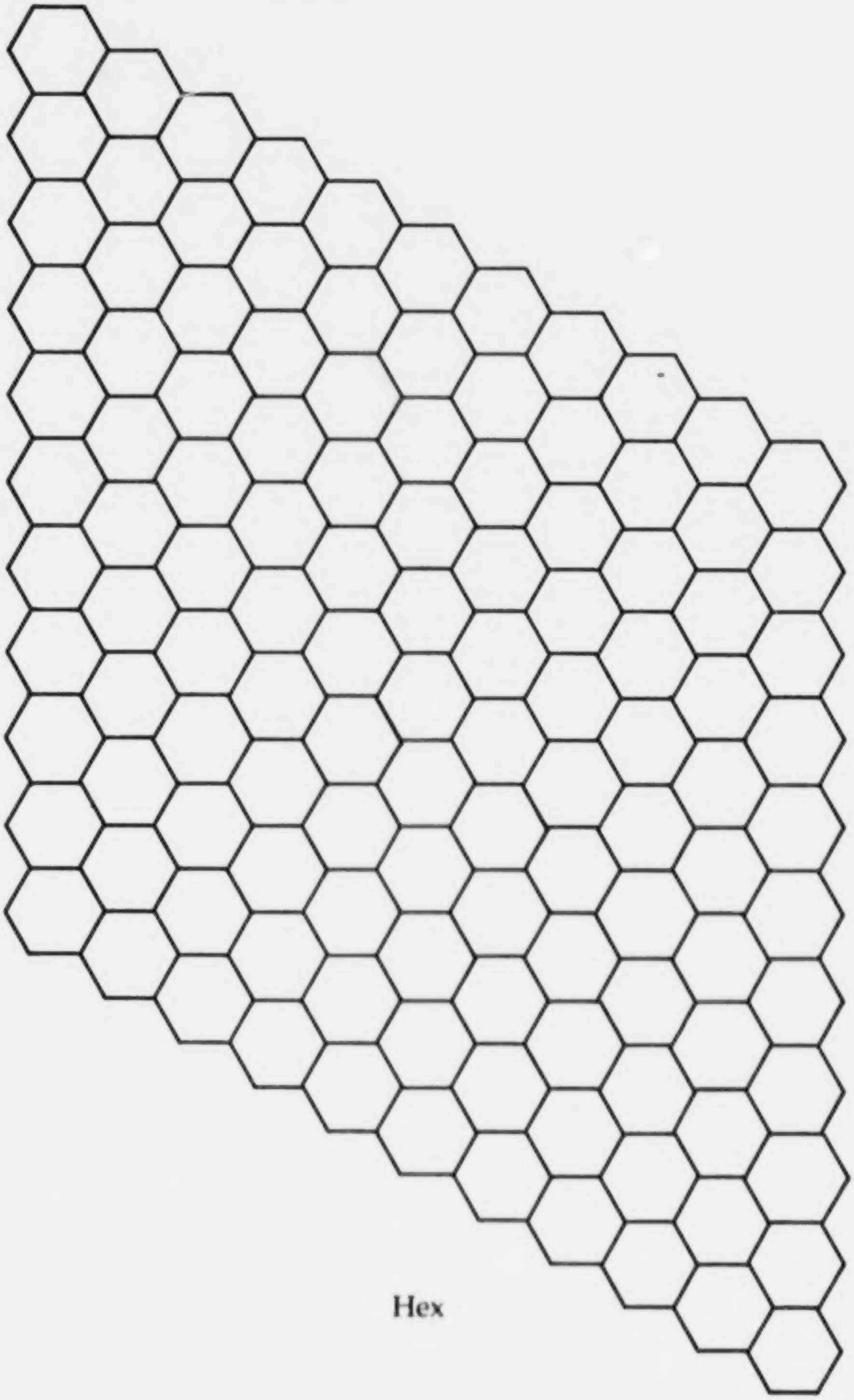

Hex

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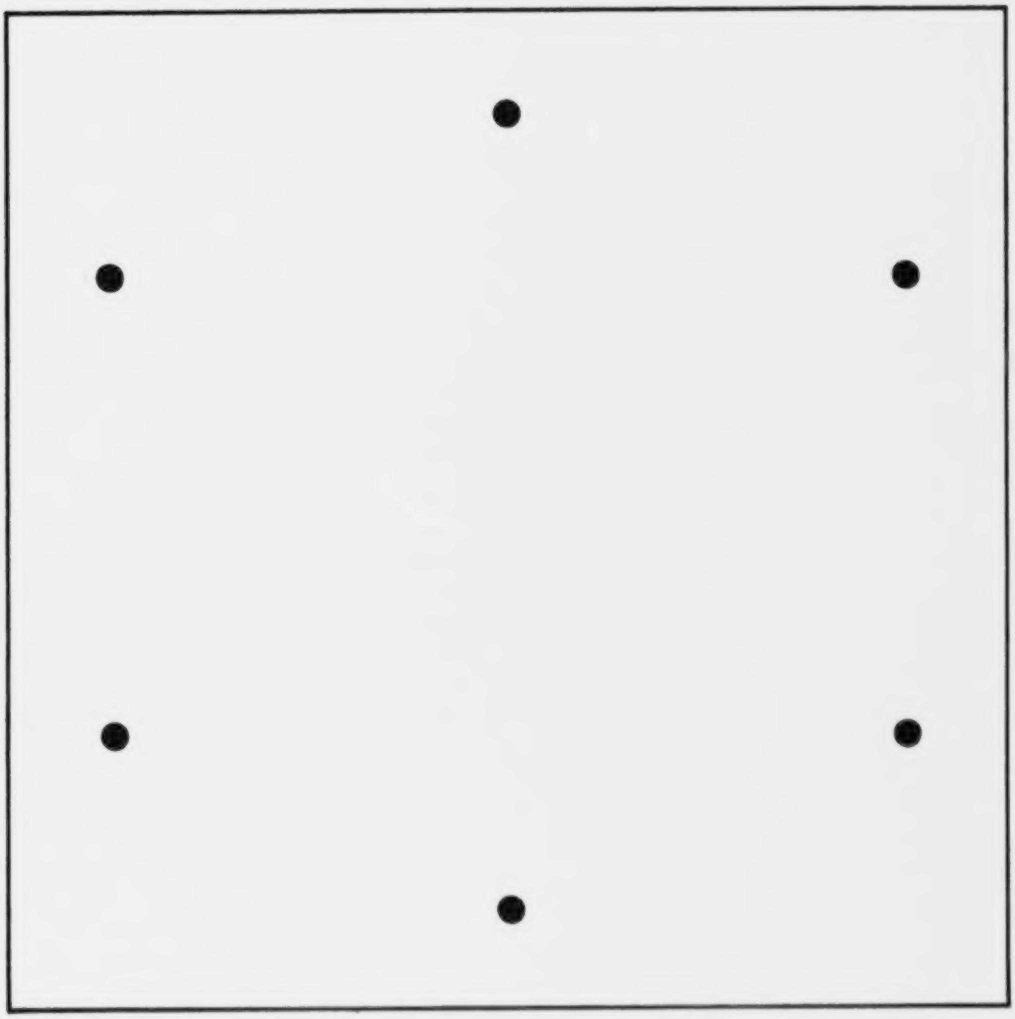

Triahex

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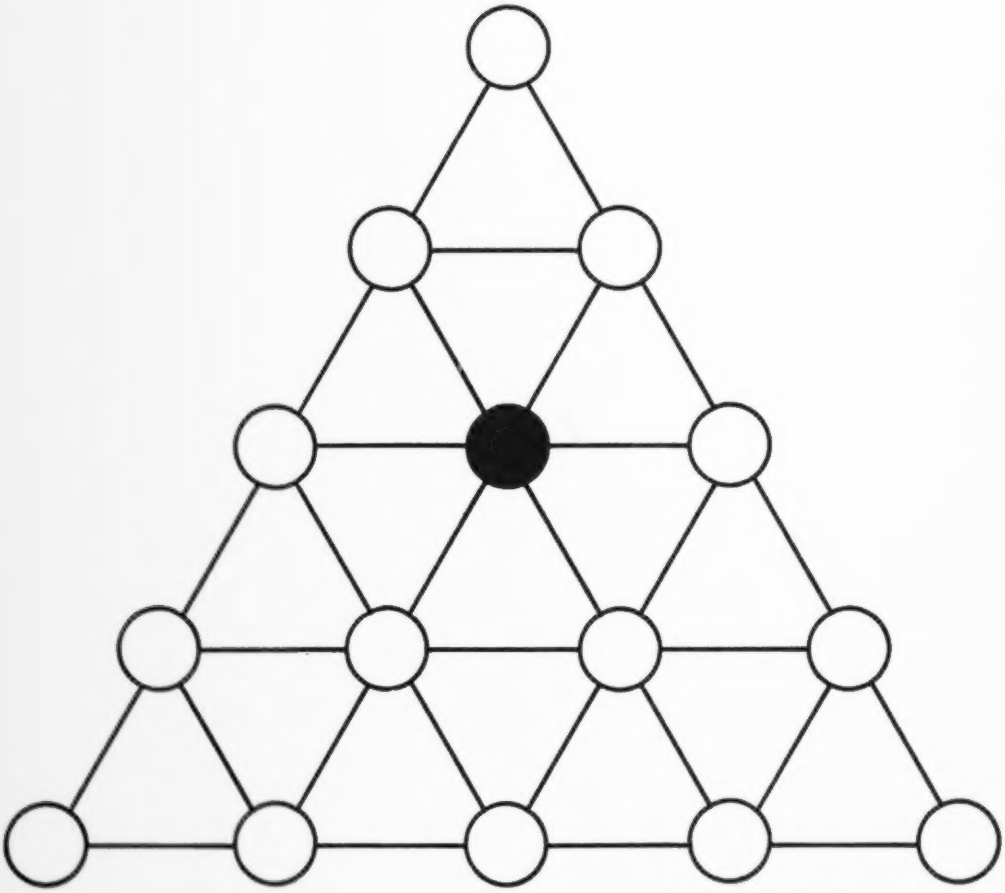

Solitaire

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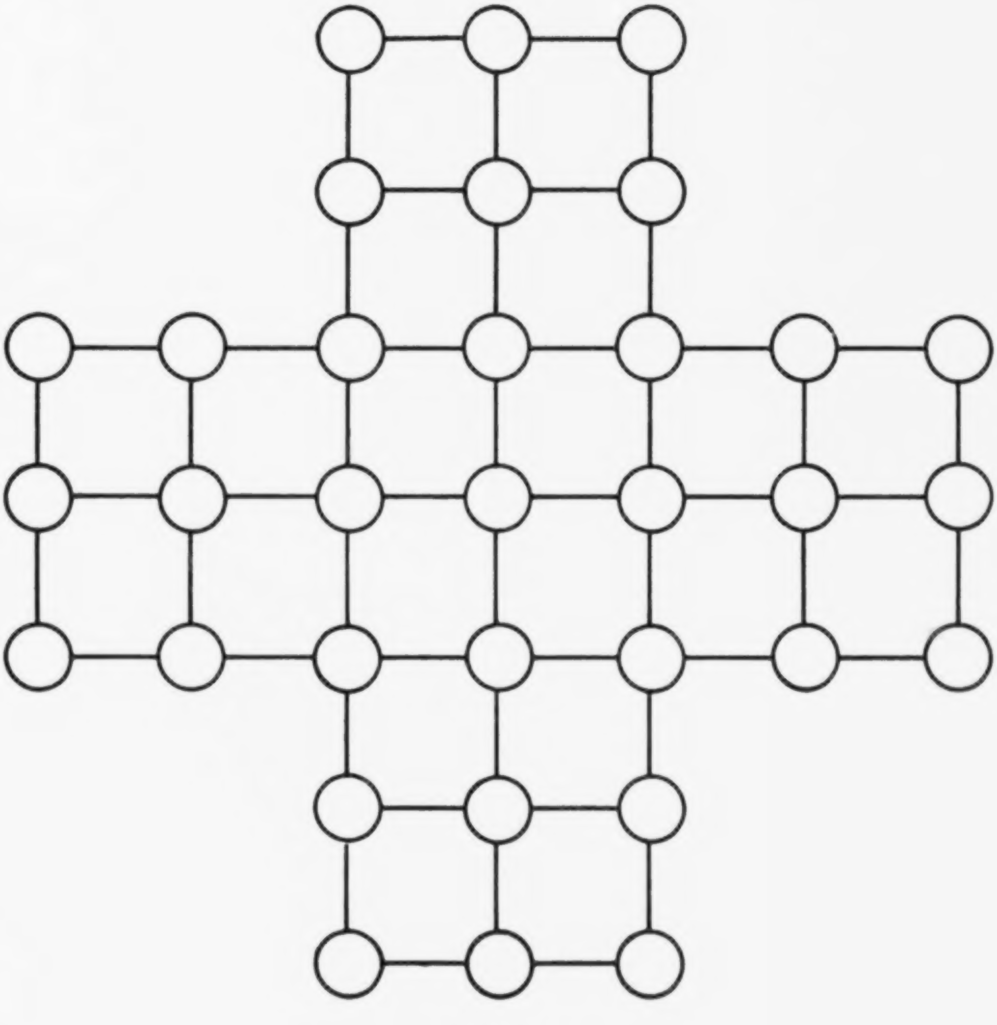

Fox and Geese

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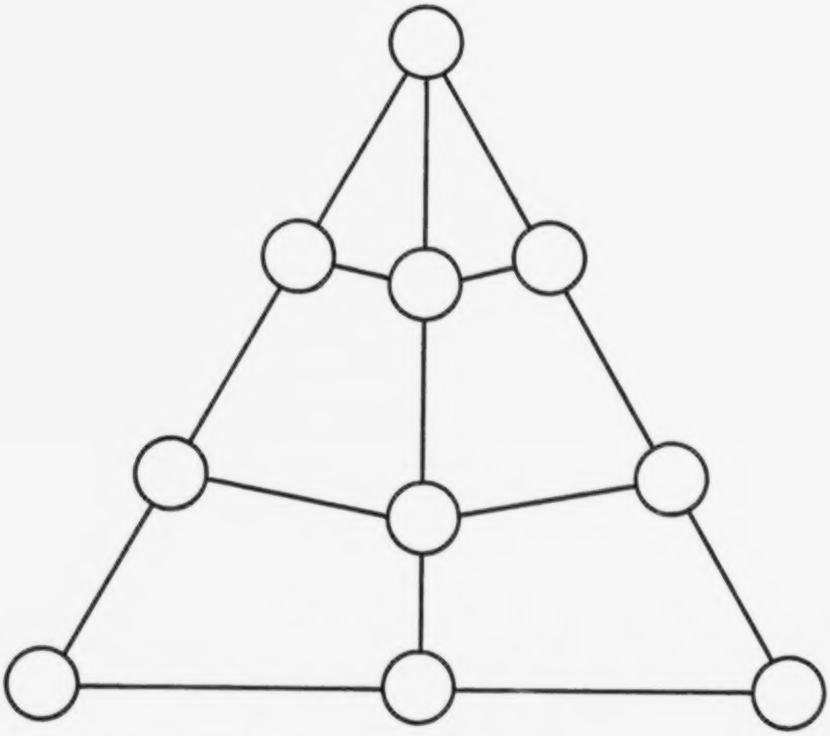

The Wolf and the Farmers

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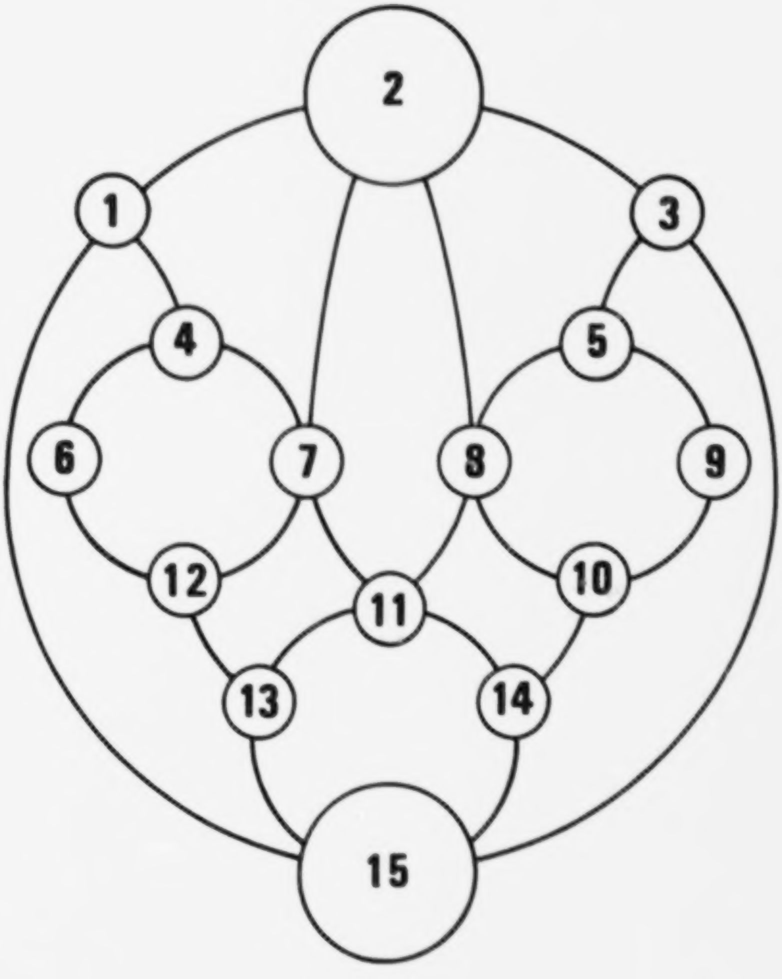
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